Supplemental Appendix for

Leaning without Sowing the Wind: Optimal Monetary Policy in Speculative Booms

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Appendix A: General model

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A. General model

This appendix presents a general version of our model that nests the benchmark model (Section 1) and several extensions. We also provide derivations of the results in Section 1 of the main text. The empirical relevance of the zero lower bound (ZLB) is discussed in Appendix A.7.

A.1 States and transition rates: Aggregate shocks and monetary policy

States are denoted by $s \in S$. States can differ in terms of the exogenous growth rate of capital, g_s, and monetary policy. The beliefs of type $i \in I \equiv \{o, p, pl\}$ about the transition rate from s to $s' \neq s$ are denoted by $\lambda^i_{s\to s'}$. We assume that for every s and $s' \neq s$, either $\lambda^i_{s\to s'} > 0$ for all $i \in I$ or $\lambda^i_{s\to s'} = 0$ for all $i \in I$. From any state s, there may be zero, one, or two states $s' \neq s$ with $\lambda^i_{s\to s'} > 0$. For any state s, there is at most one state $s' \neq s$ such that $\lambda^o_{s\to s'} \neq \lambda^p_{s\to s'}$.

In Section 3, to study formally the adoption of leaning monetary policy, we study a model in which $S \equiv \{0, 1, 2, 3\}$. In Section 5.2, we add a non-speculative boom $s = 1'$ to allow some speculative booms to end without a bust. In Section 5.3, to expand the model beyond one boom-bust cycle, $S \equiv \{1, 2, 3, 4, 5\}$. That is, to analyze systematic policy, we drop state 0 from the model of Section 3. We then modify the resulting model in the simplest way to obtain more than one boom-bust cycle, by adding a state 3 boom and a state 4 bust, with recovery in state 5. The recurrent-cycle extension features $S \equiv \{1, 2\}$. Cycles are recurrent in that $\lambda^i_{1\to 2} > 0$ and $\lambda^i_{2\to 1} > 0$. Figure IA.1 illustrates the state structure.

A.2 Investors' problem: Portfolio choice and consumption

We begin the analysis for the case in which each state s either has a single successor state s' with $\lambda^i_{s\to s'} > 0$ or is an absorbing state. (The model in which a state can have two successor states is presented in Appendix E.2). Investors' problem is given by Eqs. (3)–(5). Define the (net) return on the risky asset upon transition from s to s' at time t as $r_{t,s\to s'} \equiv \frac{Q_{t,s'}}{Q_{t,s}} - 1$. The Hamilton-Jacobi-Bellman (HJB) equation for an investor with

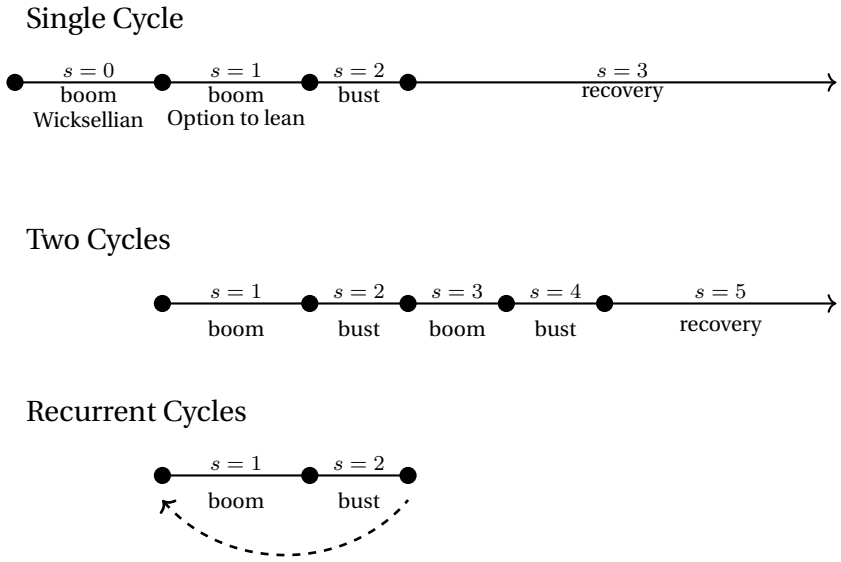


Figure IA.1: **State sequences**. See Appendix A.1.

belief type i and wealth $a^i_{t,s}$ at time t in state s is:

$$\rho V^i_{t,s}(a^i_{t,s}) = \max_{c,\omega} \log(c) + \frac{\partial V^i_{t,s}}{\partial a}\left(a^i_{t,s}\left(r^f_{t,s} + \omega(r_{t,s} - r^f_{t,s})\right) - c\right) + \frac{\partial V^i_{t,s}(a^i_{t,s})}{\partial t} + \lambda^i_{s\to s'}\left(V^i_{t,s'}\left(a^i_{t,s}\left(1 + \omega r_{t,s\to s'}\right)\right) - V^i_{t,s}(a^i_{t,s})\right) \tag{IA.1}$$

subject to $\omega \leq \bar{\omega}_s$. Log utility implies that two infinitesimal investors with the same belief type i but different wealth at time t ($a^{i,h}_{t,s}$ and $a^{i,l}_{t,s}$, with $a^{i,h}_{t,s} > a^{i,l}_{t,s}$ and where h and l denote "high" and "low" wealth, respectively) will, at every future date $t' \geq t$ and each state $s' \in S$, for every possible realization of aggregate shocks (i.e., timing of state transitions), have a constant ratio of consumption: $\frac{c^{i,h}_{t',s'}}{c^{i,l}_{t',s'}} = \frac{a^{i,h}_{t,s}}{a^{i,l}_{t,s}}$. Correspondingly, lifetime utility has the form:

$$V^i_{t,s}(a^i_{t,s}) = \frac{1}{\rho}\log(\frac{a^i_{t,s}}{Q_{t,s}}) + \underbrace{V^i_{t,s}(Q_{t,s})}_{v^i_{t,s}}, \tag{IA.2}$$

where $v^i_{t,s}$ is defined as the lifetime utility (expected discounted utility of consumption) of a type-i investor with wealth $Q_{t,s}$, the price of a unit asset position.[1] Differentiating Eq.

[1]To verify Eq. (IA.2), one may guess that $V^i_{t,s}(a) = A + B\log(a)$, where A and B are unknown coefficients. Substituting this conjecture into the HJB (IA.1), taking the first-order conditions, and re-substituting the result into the HJB, one obtains Eq. (IA.2).

(IA.2) with respect to $a^i_{t,s}$, one obtains:

$$V^i_{a,t,s}(a^i_{t,s}) \equiv \frac{\partial V^i_{t,s}(a)}{\partial a}|_{a=a^i_{t,s}} = \frac{1}{\rho a^i_{t,s}}, \tag{IA.3}$$

where $V^i_{a,t,s}$ is the marginal lifetime utility of wealth. The first-order condition for consumption c equates the marginal utility of consumption with the marginal lifetime utility of wealth

$$\frac{1}{c^i_{t,s}} = \frac{1}{\rho a^i_{t,s}} \quad \text{or} \quad c^i_{t,s} = \rho a^i_{t,s}. \tag{IA.4}$$

The first-order condition for ω implies

$$\underbrace{r_{t,s}}_{\substack{\text{Return on risky asset}\\ \text{absent transition}}} + \underbrace{\lambda^i_{s\to s'} \frac{V^i_{a,t,s'}(a^i_{t,s'})}{V^i_{a,t,s}(a^i_{t,s})}}_{\substack{\text{Stochastic}\\ \text{discount factor}}} \underbrace{r_{t,s\to s'}}_{\substack{\text{Return on}\\ \text{risky asset}\\ \text{upon}\\ \text{state transition}}} \geq \underbrace{r^f_{t,s}}_{\text{Risk-free rate}} \tag{IA.5}$$

where $\lambda^i_{s\to s'} \frac{V^i_{a,t,s'}(a^i_{t,s'})}{V^i_{a,t,s}(a^i_{t,s})}$ is the stochastic discount factor of an investor (with belief type i) for transition from s to s'. This stochastic discount factor is the perceived arrival rate of transition multiplied by the ratio of the marginal lifetime utility of wealth if the event occurs immediately, relative to the marginal lifetime utility of wealth in the current state (absent state transition). Eq. (IA.5) holds with equality if $\omega < \bar{\omega}_s$. Substituting Eq. (IA.4) into Eq. (IA.5),

$$r_{t,s} + \lambda^i_{s\to s'} \frac{a^i_{t,s}}{a^i_{t,s'}} r_{t,s\to s'} \geq r^f_{t,s}. \tag{IA.6}$$

Substituting $a^i_{t,s} = \alpha^i_{t,s} Q_{t,s} k_{t,s}$, one obtains Eq. (14).

A.3 Equilibrium in the goods market and asset market

Output-asset price relation. Each investor's consumption is a constant fraction ρ of the investor's wealth, as shown by Eq. (IA.4), implying the third equality in Eq. (9) in the main text.

Utilization absent nominal rigidities. The first-order condition of Eq. (7) is $\delta'(\eta_{t,s})Q_{t,s} = 1$. Substituting $\eta_{t,s} = \eta^*$ and $Q_{t,s} = Q^* = \frac{\eta^*}{\rho}$ (from the output-asset price relation, Eq. (9)), we then obtain Eq. (11) as a characterization of utilization and the asset price absent nominal rigidities.

Law of motion for wealth shares absent state transition. From Eq. (12),

$$\frac{\dot{\alpha}^i_{t,s}}{\alpha^i_{t,s}} = \frac{\dot{a}^i_{t,s}}{a^i_{t,s}} - \frac{\frac{d[Q_{t,s}k_{t,s}]}{dt}}{Q_{t,s}k_{t,s}}. \tag{IA.7}$$

Combining Eq. (IA.4) and the budget constraint Eq. (4), one obtains

$$\frac{\dot{a}^i_{t,s}}{a^i_{t,s}} = r^f_{t,s} + \omega^i_{t,s}(r_{t,s} - r^f_{t,s}) - \rho.$$

Using also the market clearing condition $\omega^o_{t,s}a^o_{t,s} + \omega^p_{t,s}a^p_{t,s} = Q_{t,s}k_{t,s}$, we obtain

$$\frac{\frac{d[Q_{t,s}k_{t,s}]}{dt}}{Q_{t,s}k_{t,s}} = r_{t,s} - \rho. \tag{IA.8}$$

The growth rate of aggregate wealth absent transition equals the return on capital absent transition less the consumption share of wealth. Eqs. (IA.7)–(IA.8) imply

$$\frac{\dot{\alpha}^i_{t,s}}{\alpha^i_{t,s}} = \left(\omega^i_{t,s} - 1\right)\left(r_{t,s} - r^f_{t,s}\right).$$

Substituting from Eq. (14) (for $i = p$, because this expression holds with equality for pessimists) and Eq. (13), one obtains Eq. (15).

If the leverage constraint does not bind for either investor type, then Eq. (14) holds with equality for both investors. This implies $\frac{\lambda^o_{s\to s'}}{\lambda^p_{s\to s'}} = \frac{\alpha^o_{t,s'}}{\alpha^o_{t,s}}\frac{\alpha^p_{t,s}}{\alpha^p_{t,s'}}$. Using this expression with $\alpha^o_{t,s} = 1 - \alpha^p_{t,s}$ and $\alpha^o_{t,s'} = 1 - \alpha^p_{t,s'}$, we obtain the first expression in Eq. (16). Using this result together with Eq. (15), we obtain the second expression in Eq. (16). Substituting this second expression into (14), we obtain Eq. (17).

Neutral rates. Let s be a recession state and let s' be a successor state to s with $\lambda^o_{s\to s'} > \lambda^p_{s\to s'} > 0$.[2] To guarantee a negative neutral rate during the recession, we assume $\delta(0) - (\rho + \lambda^p_{s\to s'}) < g_s < \delta(0) - \rho$.

Let s be a boom state and let s' be a successor state with $\lambda^p_{s\to s'} > \lambda^o_{s\to s'} > 0$. To ensure that the neutral rate is strictly positive during the boom states, we assume $\rho + g_s - \delta(\eta^*) - \lambda^p_{s\to s'}(\frac{Q^*}{Q_{s'}(0)} - 1) > 0$. Similarly, in a recovery (absorbing) state s, we assume $\rho + g_s - \delta(\eta^*) > 0$.

Leaning monetary policy in booms. Let s be a recession state with successor state s'. The central bank sets the risk-free rate to achieve asset price $Q_{s'}(\alpha, \underline{\boldsymbol{\alpha}})$ during state s' when optimists' wealth share in state s' is α, conditional on optimists' wealth share just

[2] In our $s \in \{0,1,2,3\}$ economy, this setup characterizes $s = 2$ and $s' = 3$. In the $s \in \{1,2,3,4,5\}$ economy, this setup characterizes $(s,s') \in \{(2,3),(4,5)\}$. In the recurrent cycles, this setup characterizes $s = 2$ and $s' = 1$.

prior to the end of state s equal to $\underline{\boldsymbol{\alpha}}$. Define $\underline{Q}_{s'}(\alpha) \equiv Q_{s'}(\alpha_{s\to s'}(\alpha),\alpha) = Q_{s'}(\frac{\lambda^o_{s\to s'}}{\bar{\lambda}_{s\to s'}(\alpha)}\alpha,\alpha)$, where the equality follows from Eq. (16) and $\bar{\lambda}_{s\to s'}(\alpha) = \alpha\lambda^o_{s\to s'} + (1-\alpha)\lambda^p_{s\to s'}$. Conditional on the Q-targeting rule $Q_{s'}(\cdot)$ and optimists' wealth share just prior to the end of state s equal to α, the asset price upon transition to s' is $\underline{Q}_{s'}(\alpha)$.

We restrict attention to Q-targeting rules, $Q_{s'}(\cdot)$, that are right-differentiable and satisfy $\frac{d\underline{Q}_{s'}(\alpha)}{d\alpha} \geq 0$. This assumption is very general. First, this assumption is satisfied by a Wicksellian targeting rule, $Q_{s'}(\alpha,\underline{\boldsymbol{\alpha}}) = Q^*$ for $(\alpha,\underline{\boldsymbol{\alpha}}) \in [0,1]\times[0,1]$. Second, in the $S=\{0,1,2,3\}$ economy, this assumption imposes no restrictions on leaning monetary policy during the $s=1$ boom (because $s=1$ is not the successor to a recession). Third, in multi-cycle economies, this assumption is satisfied if the central bank follows a Q-targeting rule $Q_{s'}(\alpha,\underline{\boldsymbol{\alpha}})$ that does not depend on optimists' wealth share prior to the boom (i.e., $\frac{dQ_{s'}(\alpha,\underline{\boldsymbol{\alpha}})}{d\underline{\boldsymbol{\alpha}}} = 0$) and with a risky asset price that is weakly increasing over time during booms (i.e., $\frac{dQ_{s'}(\alpha,\underline{\boldsymbol{\alpha}})}{d\alpha} \geq 0$).[3] A further class of rules consistent with this assumption are rules with $Q_{s'}(\alpha_{s\to s'}(\alpha),\alpha) = Q^*$ (or equal to any other constant). This class includes the optimal policy characterized in Sections 4 and 5.3.

In Section 1, we presented the *cure-no-worse-than-the-disease* property: $Q_1(\alpha) \geq Q_2(\alpha)$, for $\alpha\in[0,1]$. A more general definition of this property is as follows. Consider a boom state s with successor state s'. An s-state targeting rule $Q_s(\cdot)$ satisfies the cure-no-worse-than-the-disease property if $Q_s(\alpha,\underline{\boldsymbol{\alpha}}) \geq Q^d_{s'}(\alpha)$, where $Q^d_{s'}(\alpha)$ is the solution to the risk-balance condition:

$$\rho + g_{s'} - \delta\left(\frac{Q^d_{s'}(\alpha)}{Q^*}\eta^*\right) + \frac{d\ln Q^d_{s'}(\alpha)}{d\alpha}\dot{\alpha}_{s'}(\alpha) + \bar{\lambda}_{s'\to s''}(\alpha)\left(1-\frac{Q^d_{s'}(\alpha)}{Q^*}\right) = 0, \tag{IA.9}$$

where s'' is the successor state to s'. In the applications in this paper, s' is a bust state in which leverage constraints are slack, so $\dot{\alpha}_{s'}(\alpha)$ is given by Eq. (16) and depends only on beliefs and α (and not on $Q^d_{s'}$). The boundary condition is Eq. (IA.9) applied to $\alpha=0$.

A.4 Risky asset price during bust states

Let s be a bust state with successor state s' and $\lambda^o_{s\to s'} > \lambda^p_{s\to s'} > 0$. In our $s\in\{0,1,2,3\}$ economy, we have $(s,s') = (2,3)$. In the $s\in\{1,2,3,4,5\}$ economy, $(s,s')\in\{(2,3),(4,5)\}$. With recurrent cycles, $(s,s') = (2,1)$. Our results are for the case $\bar{\omega}_s = \infty$. (For the

[3] This class of rules includes the simple (non-optimal) policy in Eq. (28), used in our numerical analyses of simple rules.

successor state s', we allow for a leverage limit $\bar{\omega}_{s'} \in (1,\infty)$ or $\bar{\omega}_{s'} = \infty$.)

Lemma IA.1 *The asset price in the bust satisfies* $Q_s(\alpha) < \underline{Q}_{s'}(\alpha) \leq Q^*$ *and* $\frac{dQ_s(\alpha)}{d\alpha} > 0$ *for* $\alpha \in (0,1)$.

Proof of Lemma IA.1. Consider $\alpha \in \{0,1\}$. In these boundary cases, $\dot{Q}_{t,s} = 0$ (the bust asset price is at a steady state). Denote the asset price in the economy with only $i \in \{o,p\}$ beliefs by Q_s^i. From Eqs. (10) and (17) and the conjecture $r_s^f = 0$,

$$\rho + g_s - \delta(\frac{Q_s^i}{Q^*}\eta^*) + \lambda_{s\to s'}^i(1 - \frac{Q_s^i}{\underline{Q}_{s'}^i}) = 0, \tag{IA.10}$$

where $\underline{Q}_{s'}^i$ is the asset price upon recovery ($\underline{Q}_{s'}^p = \underline{Q}_{s'}(0)$ and $\underline{Q}_{s'}^o = \underline{Q}_{s'}(1)$). Under the parametric assumptions (see "Neutral rates" above), we can show that there exists a unique $Q_s^i \in (0, \underline{Q}_{s'}^i)$ satisfying Eq. (IA.10). The left-hand side: is strictly positive, when evaluated at $Q_s^i = 0$; is strictly negative, when evaluated at $Q_s^i = \underline{Q}_{s'}^i$; and is continuous and decreasing in Q_s^i. To verify that the expression is strictly negative when evaluated at $Q_s^i = \underline{Q}_{s'}^i$, note

$$\rho + g_s - \delta(\frac{\underline{Q}_{s'}^i}{Q^*}\eta^*) \leq \rho + g_s - \delta(0) < 0,$$

where the first inequality follows because $\delta(\eta)$ is weakly increasing in η. The second inequality follows from the parametric assumptions.

We can also show that $Q_s^o > Q_s^p$. The left-hand side of Eq. (IA.10) is strictly decreasing in Q_s^i, strictly increasing in $\lambda_{s\to s'}^i$ (for $Q_s^i < \underline{Q}_{s'}^i$), and strictly increasing in $\underline{Q}_{s'}^i$. Therefore, from $\lambda_{s\to s'}^o > \lambda_{s\to s'}^p$ and the assumption that $\underline{Q}_{s'}(\alpha)$ is weakly increasing, we have $Q_s^o > Q_s^p$.

The remainder of the proof follows similar steps to Section B.3 of the online appendix for Caballero and Simsek (2020b), with the main difference being that the asset price upon recovery there is Q^* whereas here it is $\underline{Q}_{s'}(\alpha)$, which is a policy choice of the central bank (a part of its Q-targeting rule).[4] Define $q_{t,s} = \log(Q_{t,s})$. Using Eqs. (10) and (17), we obtain the differential equation system

$$\dot{q}_{t,s} = -\left(\rho + g_s - \delta(\frac{\exp(q_{t,s})}{Q^*}\eta^*) + \bar{\lambda}_{s\to s'}(\alpha_{t,s})\left(1 - \frac{\exp(q_{t,s})}{\underline{Q}_{s'}(\alpha_{t,s})}\right)\right) \tag{IA.11}$$

[4]There are other differences including the inclusion of the depreciation function $\delta(\eta)$ here as well as a different menu of financial assets in Caballero and Simsek (2020b) to accommodate additional types of shocks while maintaining dynamically complete markets.

and

$$\dot{\alpha}_{t,s} = -\alpha_{t,s}(1-\alpha_{t,s})(\lambda^o_{s\to s'} - \lambda^p_{s\to s'}). \tag{IA.12}$$

We analyze this system using the phase diagram in Figure IA.2. For a given $Q_{s'}(\cdot)$ policy, the $\dot{q}_{t,s} = 0$ locus is $q^h_s(\alpha_{t,s})$ defined implicitly by:

$$\rho + g_s - \delta(\frac{\exp(q^h_s(\alpha_{t,s}))}{Q^*}\eta^*) + \bar{\lambda}_{s\to s'}(\alpha_{t,s})\left(1 - \frac{\exp(q^h_s(\alpha_{t,s}))}{\underline{Q}_{s'}(\alpha_{t,s})}\right) = 0, \tag{IA.13}$$

where $\bar{\lambda}_{s\to s'}(\alpha_{t,s}) = \alpha_{t,s}\lambda^o_{s\to s'} + (1-\alpha_{t,s})\lambda^p_{s\to s'}$. Comparing (IA.13) with Eq. (IA.10) for the bust asset price with $\alpha \in \{0,1\}$, we find $q^h_s(0) = \log(Q^p_s)$ and $q^h_s(1) = \log(Q^o_s)$. This observation and Eq. (IA.12) imply that the steady states of the system Eq. (IA.11)–(IA.12) are $(\alpha, q_s) = (0, \log(Q^p_s))$ and $(\alpha, q_s) = (1, \log(Q^o_s))$. As can be seen from the phase diagram, in this system, for any $\alpha_{t,s} \in (0,1)$, optimists' wealth share and log asset price will converge (absent state transition) to the $(\alpha, q_s) = (0, \log(Q^p_s))$ steady state.

Using Eq. (IA.13), there exists a unique $\exp(q^h_s(\alpha)) \in (0, \underline{Q}_{s'}(\alpha))$ for $\alpha \in (0,1)$. Similar to the preceding argument for common-belief economies, the left-hand side of Eq. (IA.13) is: strictly positive, in the limit as $\exp(q^h_s(\alpha)) \to 0$; strictly negative, when evaluated at $\exp(q^h_s(\alpha)) = \underline{Q}_{s'}(\alpha)$; and continuous and decreasing in $\exp(q^h_s(\alpha))$.

Differentiating (IA.13) with respect to α, we have

$$\begin{aligned}(\lambda^o_{s\to s'} - \lambda^p_{s\to s'})\left[1 - \frac{\exp(q^h_s(\alpha))}{\underline{Q}_{s'}(\alpha)}\right] + \bar{\lambda}_{s\to s'}(\alpha)\frac{\exp(q^h_s(\alpha))}{\underline{Q}_{s'}(\alpha)^2}\frac{d\underline{Q}_{s'}(\alpha)}{d\alpha} = \\ \left[\delta'\left(\frac{\exp(q^h_s(\alpha))}{Q^*}\eta^*\right)\frac{\exp(q^h_s(\alpha))}{Q^*}\eta^* + \bar{\lambda}_{s\to s'}(\alpha)\frac{\exp(q^h_s(\alpha))}{\underline{Q}_{s'}(\alpha)}\right]\frac{dq^h_s(\alpha)}{d\alpha}\end{aligned} \tag{IA.14}$$

Consider the term in brackets that multiplies $\frac{dq^h_s(\alpha)}{d\alpha}$ on the second line. This term is strictly positive because $\delta'(\eta) \geq 0$ for $\eta \in [0, \eta^*]$, $\underline{Q}_{s'}(\alpha) > 0$, and $\bar{\lambda}_{s\to s'}(\alpha) > 0$. Earlier in this proof, we showed that $q^h_s(\alpha) < \log(\underline{Q}_{s'}(\alpha))$. We also have $\lambda^o_{s\to s'} > \lambda^p_{s\to s'}$ (optimists perceive a higher arrival rate of the subsequent boom than pessimists do). From $\frac{d\underline{Q}_{s'}(\alpha)}{d\alpha} \geq 0$ (see "Leaning monetary policy in booms"), we obtain that $q^h_s(\alpha)$ is increasing in α, as shown in Figure IA.2. With the assumption that $\underline{Q}_{s'}(\alpha)$ is right-differentiable, the system (IA.11)–(IA.12) satisfies the Lipschitz condition and is saddle path stable, with $q_{t,s} = q_s(\alpha_{t,s})$, where $q_s(\alpha)$ is increasing in α.

Eqs. (IA.11)–(IA.12) can be written as a differential equation in the α domain. In the

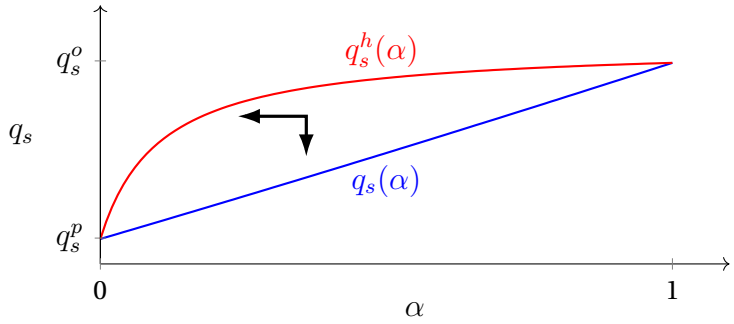


Figure IA.2: **Phase diagram for equilibrium in bust state** s. Regarding optimists' wealth share, $\dot{\alpha}_{t,s} < 0$ for $\alpha \in (0,1)$. Regarding the (log) asset price, $\dot{q}_{t,s} < 0$ if and only if $q_s < q_s^h(\alpha)$.

case of Wicksellian policy in the successor state following recession ($\underline{Q}_{s'}(\alpha) = Q^*$), we have:

$$\frac{d\ln Q_s(\alpha)}{d\alpha} = \frac{1}{\alpha(1-\alpha)(\lambda^o_{s\to s'} - \lambda^p_{s\to s'})}\left(\rho + g_s - \delta\left(\frac{Q_s(\alpha)}{Q^*}\eta^*\right) + \bar{\lambda}_{s\to s'}(\alpha)\left(1 - \frac{Q_s(\alpha)}{Q^*}\right)\right)$$

and the boundary conditions given by Eq. (IA.10) with $\underline{Q}^i_{s'} = Q^*$.

Proof of Lemma 1. The first claims in Lemma 1 follow immediately from Lemma IA.1. With $\delta(\eta) = \bar{\delta}$, Eq. (18) follows from Proposition IA.1 in Appendix B (with $\chi = 0$).

Leverage limit. We assume $\bar{\omega} < \hat{\omega}$, where $\hat{\omega} = \frac{1}{1-\frac{Q_2(0)}{2Q^*}}$. This assumption requires that the leverage limit is below a threshold $\hat{\omega}$, with this threshold strictly greater than 1 (because $0 < Q_2(0) < 2Q^*$) and increasing in $Q_2(0)/Q^*$, or, the ratio of the all-pessimist recession asset price to the efficient asset price. This assumption (a sufficient condition that can be relaxed) is used to prove the uniqueness of $\alpha_{0\to 1}(\alpha)$ in the proof of Proposition 1.[5]

Leaning monetary policy in $s = 1$ generates a macroprudential benefit: When the leverage constraint is binding, a lower asset price during the boom reduces the severity of the subsequent recession (Caballero and Simsek 2020a), *conditional on optimists' wealth share prior to the arrival of recession*, as explained in the next result, Lemma IA.2.

Lemma IA.2 *Suppose* $Q_1(\alpha) > Q_2(\alpha)$ *(i.e., cure-no-worse-than-the-disease holds strictly). Upon transition from boom state* $s \in \{0,1\}$ *to a recession* $s = 2$*, optimists' wealth share*

[5]The assumption $\bar{\omega} < \hat{\omega}$ does not hold in our calibrated model, but we verify numerically that $\alpha_{0\to 1}(\alpha)$ is unique; see main-text Figure 2.

falls: $\alpha_{s\to 2}(\alpha) \leq \alpha$, *for* $s \in \{0,1\}$ *and* $\alpha \in (0,1)$. *Specifically,*

$$\alpha_{s\to 2}(\alpha) = \begin{cases} \alpha \frac{\lambda^o_{s\to 2}}{\bar{\lambda}_{s\to 2}(\alpha)} & \text{if } \omega^o_s(\alpha) < \bar{\omega} \\ \alpha\left[1-(\bar{\omega}-1)\left(\frac{Q_s(\alpha)}{Q_2(\alpha_{s\to 2}(\alpha))}-1\right)\right] & \text{if } \omega^o_s(\alpha) = \bar{\omega}, \end{cases} \tag{IA.15}$$

with $\frac{d\alpha_{s\to 2}(\alpha)}{dQ_s(\alpha)} \leq 0$, *and strictly so if and only if the leverage limit is binding. There is a threshold* $Q^{bind}_s(\alpha)$ *such that the leverage constraint binds if and only if* $Q_s(\alpha) \leq Q^{bind}_s(\alpha)$.

Lemma IA.2 demonstrates a macroprudential benefit of leaning. An associated cost, made clear in Eq. (15), is that a targeting rule with larger macroprudential benefits (higher $\alpha_{1\to 2}(\alpha)$) implies a slower growth rate of optimists' wealth share (lower $\dot{\alpha}_1(\alpha)$).

Proof of Lemma IA.2. The top line of Eq. (IA.15) follows immediately from Eq. (16). The bottom line follows from Eq. (13). Finally, using (13) and the top line of (IA.15), the leverage constraint binds for optimists if and only if $Q_s(\alpha) \leq Q^{bind}_s(\alpha)$, defined as the solution to

$$\frac{\lambda^o_{s\to 2}}{\bar{\lambda}_{s\to 2}(\alpha)} = 1-(\bar{\omega}-1)\left(\frac{Q^{bind}_s(\alpha)}{Q_2(\frac{\lambda^o_{s\to 2}}{\bar{\lambda}_{s\to 2}(\alpha)}\alpha)}-1\right). \tag{IA.16}$$

Next, we will show $\alpha_{s\to 2}(\alpha) < \alpha$ for $\alpha \in (0,1)$. If the leverage constraint is not binding, this result is immediate from the top line of Eq. (IA.15) and $\lambda^o_{s\to 2} < \lambda^p_{s\to 2}$. To address the case of a binding leverage constraint ($\omega^o_s(\alpha) = \bar{\omega}$), we define $f^{s\to 2}(\alpha_2, \alpha)$ as follows:

$$f^{s\to 2}(\alpha_2,\alpha) = \alpha\left[1-(\bar{\omega}-1)\left(\frac{Q_s(\alpha)}{Q_2(\alpha_2)}-1\right)\right]. \tag{IA.17}$$

For α with $\omega^o_s(\alpha) = \bar{\omega}$, we have $\alpha_{s\to 2}(\alpha) = f^{s\to 2}(\alpha_{s\to 2}(\alpha), \alpha)$. We will show that (i) $f^{s\to 2}(\alpha\frac{\lambda^o_{s\to 2}}{\bar{\lambda}_{s\to 2}(\alpha)}, \alpha) > \alpha\frac{\lambda^o_{s\to 2}}{\bar{\lambda}_{s\to 2}(\alpha)}$; (ii) $f^{s\to 2}(\alpha,\alpha) < \alpha$; and (iii) if $\alpha_2 \in (0,1)$ satisfies $f^{s\to 2}(\alpha_2,\alpha) = \alpha_2$, it holds that $f^{s\to 2}_{\alpha_2}(\alpha_2,\alpha) < 1$.

Using the definition Eq. (IA.17),

$$f^{s\to 2}(\alpha\frac{\lambda^o_{s\to 2}}{\bar{\lambda}_{s\to 2}(\alpha)},\alpha) = \alpha\left[1-(\bar{\omega}-1)\left(\frac{Q_s(\alpha)}{Q_2(\alpha\frac{\lambda^o_{s\to 2}}{\bar{\lambda}_{s\to 2}(\alpha)})}-1\right)\right].$$

Combining this expression with $Q_s(\alpha) < Q^{bind}_s(\alpha)$, and using (IA.16), we have claim (i). Next, substituting $\alpha_2 = \alpha$ into (IA.17), observe that

$$f^{s\to 2}(\alpha,\alpha) = \alpha\left[1-(\bar{\omega}-1)\left(\frac{Q_s(\alpha)}{Q_2(\alpha)}-1\right)\right]. \tag{IA.18}$$

Claim (ii) immediately follows from (IA.18), $Q_2(\alpha) < Q_s(\alpha) \leq Q^*$, and $\bar{\omega} > 1$. For claim

(iii), using (IA.17), we have

$$f_{\alpha_2}^{s\to 2}(\alpha_2,\alpha)=\alpha(\bar{\omega}-1)\frac{Q_s(\alpha)}{Q_2(\alpha_2)^2}Q_2'(\alpha_2). \tag{IA.19}$$

We assume $\frac{Q_2'(\alpha)}{Q_2(\alpha)}<1$ (a sufficient condition that can be relaxed). The leverage-cap assumption $\bar{\omega}<\hat{\omega}$ then implies that $f_{\alpha_2}^{s\to 2}(\alpha_2,\alpha)<1$ whenever $f^{s\to 2}(\alpha_2,\alpha)=\alpha_2$.

From (IA.15), $\frac{d\alpha_{s\to 2}(\alpha)}{dQ_s(\alpha)}=0$ if the leverage constraint is not binding. Otherwise

$$\frac{d\alpha_{s\to 2}(\alpha)}{dQ_s(\alpha)}=-\frac{\alpha(\bar{\omega}-1)\frac{1}{Q_2(\alpha_{s\to 2}(\alpha))}}{1-f_{\alpha_2}^{s\to 2}(\alpha_{s\to 2}(\alpha),\alpha)}<0, \tag{IA.20}$$

where claim (iii) above guarantees $f_{\alpha_2}^{s\to 2}(\alpha_{s\to 2}(\alpha),\alpha)<1$ and the inequality in (IA.20).

Discussion of Lemma IA.2. The leverage limit has an important role in leaning. Without a leverage limit in the good state, leaning does not mitigate recessions; optimists instead take on more leverage during the boom. When the leverage limit is slack, optimists' wealth share upon recession is proportional to the ratio of optimists' belief about the bust arrival rate to the wealth-weighted average belief (Eq. (IA.15), top line). Consequently, optimists' wealth share upon recession is unaffected by a marginal change in $Q_1(\alpha)$. For the benchmark model (Section 1 of the main text), denote by $\tilde{\alpha}$ the solution to $Q_1^{bind}(\tilde{\alpha})=\psi Q_2(\tilde{\alpha})+(1-\psi)Q^*$. Under the rule $Q_1(\alpha)=\psi Q_2(\alpha)+(1-\psi)Q^*$, leaning has no macroprudential benefit for $\alpha\in[\tilde{\alpha},1]$, because the leverage constraint would be slack. Hence, in analyses of simple rules of the form Eq. (28), we assume that $Q_1(\alpha)$ rises linearly in α over $[\tilde{\alpha},1]$, with $Q_1(1)=Q^*$.

A.5 Gap value

For an absorbing state s with no successor state (and hence with $\dot{\alpha}_s=0$), the gap value under the planner's beliefs satisfies $w_s^{pl}(\alpha)=\frac{1}{\rho}W(Q_s(\alpha))$. For a state s with a unique successor state s', the gap value w_s^{pl} under the planner's beliefs satisfies

$$\rho w_s^{pl}(\alpha)=W(Q_s(\alpha))+\lambda_{s\to s'}^{pl}\left(w_{s'}^{pl}(\alpha_{s\to s'}(\alpha))-w_s^{pl}(\alpha)\right)+\frac{\partial w_s^{pl}(\alpha)}{\partial\alpha}\dot{\alpha}_s(\alpha), \tag{IA.21}$$

where $W(Q)\equiv\ln\frac{Q}{Q^*}-\frac{1}{\rho}(\delta(\frac{Q}{Q^*}\eta^*)-\delta(\eta^*))$. In the quantitative analysis, when displaying the welfare criterion, Eq. (IA.21), we show the gap value rate, or the gap value divided by the expected length of time until recovery (conditional on being in state s), in percent.[6] The gap value for a state with two successor states is provided in Appendix E.2 and

[6] In almost all of the analysis (the exception is Section 5.1), the state transition arrival rate is exogenous and hence the gap value rate equals the gap value multiplied by an exogenous constant.

is a straightforward extension of Eq. (IA.21). The gap value with recurrent cycles is characterized in Appendix E.3. In our benchmark model of Section 1, from either boom state $s \in \{0,1\}$, the expected time until recovery is $(\lambda^{pl}_{s\to 2})^{-1} + (\lambda^{pl}_{2\to 3})^{-1}$.

A.6 Risk-free rate

Eq. (21) in the main text provides the risk-free rate for $s = 1$. For completeness, in the economy of Sections 1–2, the $s = 0$ risk-free rate is:

$$r^f_0(\alpha) = \rho + g_0 - \delta(\eta^*) - \frac{(1-\alpha)\lambda^p_{0\to 2}}{1-\alpha_{0\to 2}(\alpha)}\left(\frac{Q^*}{Q_2(\alpha_{0\to 2}(\alpha))} - 1\right),$$

and, in $s = 3$, the risk-free rate is $r^f_3(\alpha) = \rho + g_3 - \delta(\eta^*)$.

A.7 Empirical relevance of ZLB: Evidence from option prices

When prevailing interest rates are higher, the likelihood of a ZLB episode in subsequent years is expected to be reduced, but it is not necessarily eliminated. Estimates based on a variety of methods suggest that, in recent decades, there has often been a substantial probability of a ZLB episode (over a horizon of a few years or more), including at times when the prevailing policy rate was meaningfully above zero. For example, using information from interest rate derivatives, Bauer and Mertens (2019) estimate that, as of September 17, 2019, when the midpoint of the federal funds target range was $2\frac{1}{8}\%$, the probability of a ZLB episode in December 2021 was 24%. Estimates from surveys of market participants and a range of models, reported in Bauer and Mertens (2019) and Christensen (2019), point to similar probabilities in 2019 of returning to the ZLB over a two- or three-year horizon.

Taking a longer view, Figure IA.3 shows the evolution of the 15$^{\text{th}}$ percentile of the interest-rate distribution, 3 years ahead, using the methodology of Mertens and Williams (2021); the data are from Federal Reserve Bank of San Francisco (2026). Prior to 2022, the calculations are based on caps and floors on LIBOR.[7] From 2022 onward, the reference rate is the Secured Overnight Financing Rate (SOFR). Because LIBOR typically exceeded the federal funds rate, Bauer and Mertens (2019) use a threshold of 75 basis points as an estimate of LIBOR if the federal funds rate returned to the ZLB. This threshold is shown by the dashed orange line. For 2022 and beyond, we use a threshold of 15 basis points,

[7] For data and a description of the methodology, see https://www.frbsf.org/research-and-insights/data-and-indicators/interest-rate-probability-distributions/.

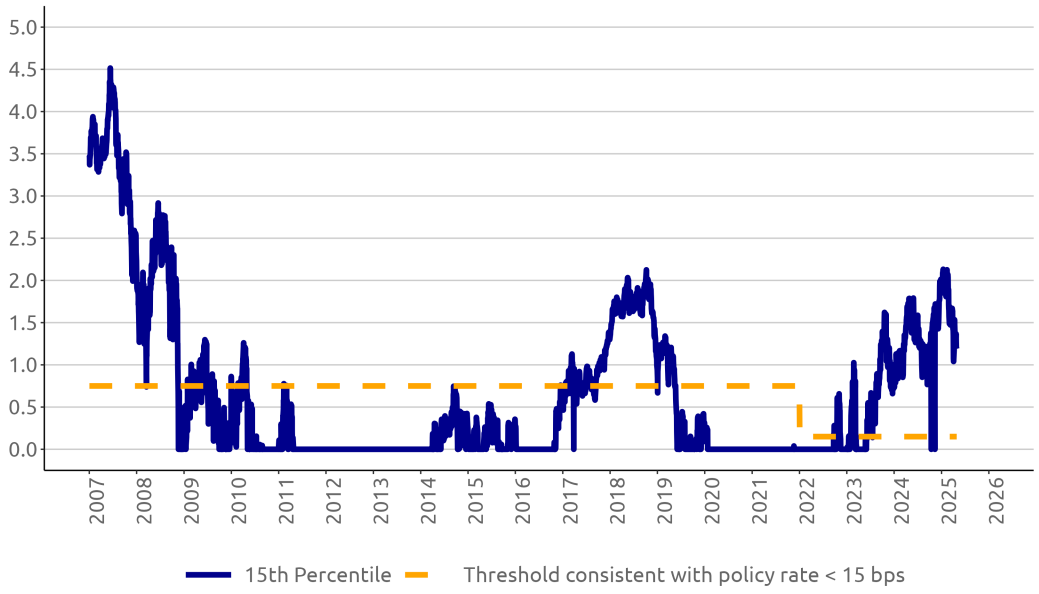


Figure IA.3: **Interest rate probability distribution, 3 years ahead**.

also shown by the dashed orange line. For much of 2022 and 2023, with the midpoint of the policy rate target range rising to $5\frac{3}{8}\%$, the estimated probability of being at the ZLB three years ahead remained above 15%.

B. A closed-form solution for the case $\delta(\eta) = \bar{\delta}$

In general, in the risk-centric model with speculation, the bust state asset price (e.g., for the one-cycle model, the solution to the differential equation Eq. (19)) does not have a closed-form solution. That is, although certain properties of the bust-state asset price can be demonstrated, the solution itself usually can only be obtained numerically.

In this section, we show, for a bust state s with successor state s', there is an intuitive closed-form solution for $Q_s(\alpha)$ if: (i) $\delta(\eta) = \bar{\delta}$ and firms choose a utilization rate $\eta_{t,s} \in [0,1]$,[8] and (ii) the Q-targeting rule for s' satisfies $\underline{Q}_{s'}(\alpha) = (1-\chi)Q^*$, where $\chi \in [0, \bar{\chi}]$. That is, $\underline{Q}_{s'}(\alpha) \in [(1-\bar{\chi})Q^*, Q^*]$, where $\bar{\chi}$ is set to ensure the cure-no-worse-than-the-disease property is satisfied. Higher values of χ correspond to more aggressive leaning, with $\chi = 0$ corresponding to Wicksellian policy.

[8]Absent nominal rigidities, firms choose $\eta^* = 1$. With nominal rigidities, output is demand-determined and firms meet any demand level with $\eta \in [0,1]$. From the output-asset price relation, the utilization rate $\eta_{t,s} = \eta^*$ is achieved when $Q_{t,s} = Q^* = 1/\rho$.

In this setting, for a bust state s, the assumptions on g_s are: $\bar{\delta} - (\rho + \lambda^p_{s\to s'}) < g_s < \bar{\delta} - \rho$ (see "Neutral rates" in Appendix A). Further, we define $Q_s^{i,Wick}$ as the asset price with only i-type investors that anticipate $\underline{Q}_{s'}(\alpha) = Q^*$, with $\lambda^i_{s\to s'}(1 - \frac{Q_s^{i,Wick}}{Q^*}) = -(\rho + g_s - \bar{\delta})$.

Proposition IA.1 (Asset price in the bust: Closed-form solution)

Suppose that $\delta(\eta) = \bar{\delta}$ and firms choose a utilization rate $\eta_{t,s} \in [0,1]$. Consider a bust state s with successor state s' with a Q-targeting rule satisfying $\underline{Q}_{s'}(\alpha) = (1-\chi)Q^*$. Then, for $\alpha \in [0,1]$,

$$\frac{1}{Q_s(\alpha)} = \frac{1}{1-\chi}\left(\frac{\alpha}{Q_s^{o,Wick}} + \frac{1-\alpha}{Q_s^{p,Wick}}\right). \tag{IA.22}$$

Note that Eq. (18) in Lemma 1 is a special case of this result, with $\chi = 0$. Eq. (41) in Section 5.3 follows from Eq. (IA.22).

Proof of Proposition IA.1. With $\delta(\eta) = \bar{\delta}$, Eq. (IA.11) becomes

$$\frac{\dot{Q}_{t,s}}{Q_{t,s}} = -\left(\rho + g_s - \bar{\delta} + \bar{\lambda}_{s\to s'}(\alpha_{t,s})\left(1 - \frac{Q_{t,s}}{\underline{Q}_{s'}(\alpha)}\right)\right). \tag{IA.23}$$

This equation is a Bernoulli equation (with exponent $n = 2$). We therefore substitute $z(t) = Q_{t,s}^{-1} = Q(t)^{-1}$ to transform (IA.23) into

$$\dot{z}(t) - \underbrace{(\bar{\rho} + \lambda^p_{s\to s'} + (\lambda^o_{s\to s'} - \lambda^p_{s\to s'})\alpha(t))}_{\equiv \upsilon(t)} z = \underbrace{-\frac{1}{(1-\chi)Q^*}(\lambda^p_{s\to s'} + (\lambda^o_{s\to s'} - \lambda^p_{s\to s'})\alpha(t))}_{\mu(t)}, \tag{IA.24}$$

where, to simplify notation, we have defined $\bar{\rho} = \rho + g_s - \bar{\delta}$ and we have dropped the state subscript s from the endogenous variables (e.g., $\alpha(t) \equiv \alpha_{t,s}$, the wealth share absent state transition). Eq. (IA.24) is a linear first-order equation that can be solved with the appropriate integrating factor. The solution is of the form:

$$z(t) = e^{\int \upsilon(t)dt}\int\left(\mu(t)e^{-\int \upsilon(t)dt}\right)dt + Ce^{\int \upsilon(t)dt}. \tag{IA.25}$$

To make progress, note that Eq. (IA.12) is a logistic equation, with

$$\alpha(t) = \frac{\alpha(0)}{(1-\alpha(0))e^{(\lambda^o_{s\to s'} - \lambda^p_{s\to s'})t} + \alpha(0)},$$

hence

$$(\lambda^o_{s\to s'} - \lambda^p_{s\to s'})\int \alpha(t)dt = \log(\alpha(t)) - \log(\alpha(0)) + (\lambda^o_{s\to s'} - \lambda^p_{s\to s'})t.$$

This result implies

$$e^{\int v(t)dt} = \frac{\alpha(t)}{\alpha(0)} \exp\left((\bar{\rho} + \lambda^o_{s\to s'})t\right). \tag{IA.26}$$

Using (IA.26), we can rewrite a key expression from (IA.25) as

$$\int \left(\mu(t)e^{-\int v(t)dt}\right) dt = -\frac{\alpha(0)}{(1-\chi)Q^*} \int \left(\frac{\lambda^p_{s\to s'}}{\alpha(t)} + (\lambda^o_{s\to s'} - \lambda^p_{s\to s'})\right) \exp\left(-(\bar{\rho} + \lambda^o_{s\to s'})t\right) dt,$$

which can be simplified further using integrals of exponential functions. After some algebra, and using the definition of $Q^{i,Wick}_s$, one obtains:

$$e^{\int v(t)dt} \int \left(\mu(t)e^{-\int v(t)dt}\right) dt = \frac{1}{(1-\chi)} \left(\frac{\alpha(t)}{Q^{o,Wick}_s} + \frac{1-\alpha(t)}{Q^{p,Wick}_s}\right). \tag{IA.27}$$

The remaining task is to determine C. We have: $\lim_{t\to\infty} Q_{t,s} = (1-\chi)Q^{p,Wick}_s$ and $\lim_{t\to\infty} \alpha_{t,s} = 0$, implying $C = 0$. Then, from Eqs. (IA.25) and (IA.27), we have Eq. (IA.22).

C. Derivations and discussion: Quantitative analysis

This appendix provides derivations and discussion for Section 2 of the main text.

C.1 Derivations

Define the risk-adjusted (net) return upon transition from s to s' for investor type i as:

$$r^{i,adj}_{t,s\to s'} = \frac{V^i_{a,t,s'}(a^i_{t,s'})}{V^i_{a,t,s}(a^i_{t,s})} r_{t,s\to s'}. \tag{IA.28}$$

where $V^i_{a,t,s}$ is the marginal utility of lifetime wealth, Eq. (IA.3). Then, the first-order condition, Eq. (IA.5), for leverage can be written as:

$$r_{t,s} + \lambda^i_{s\to s'} r^{i,adj}_{t,s\to s'} \geq r^f_{t,s}. \tag{IA.29}$$

Denote by $r^{i,adj}_{s\to s'}$ the risk-adjusted return for an investor of type i in an economy in which all investors have belief type i. That is, $r^{i,adj}_{s\to s'} = r^{i,adj}_{t,s\to s'}$ if $\alpha^i_{t,s} = 1$ (we drop the t subscript from $r^{i,adj}_{t,s\to s'}$ because the Q-targeting rule in a given state is time-invariant and $\dot{\alpha}_{t,s} = 0$ when $\alpha \in \{0,1\}$). In the bust state when all agents have belief type i, we have $\dot{Q}_{t,2} = 0$ and $r^f_{t,2} = 0$. Denote equilibrium utilization in state s when all agents have belief type i by η^i_s. Using these observations and definitions together with Eq. (IA.29) and the

definition of belief difference parameter $\xi_{s\to s'}$ in Eq. (23), with $\delta(\eta_2^o) \approx \delta(\eta_2^p)$, we have[9]

$$\xi_{2\to 3} \approx \frac{r_{2\to 3}^{p,adj} - r_{2\to 3}^{o,adj}}{\frac{1}{2}(|r_{2\to 3}^{o,adj}| + |r_{2\to 3}^{p,adj}|)}. \tag{IA.30}$$

Expression (IA.30) offers an intuitive connection between: (i) the highest and lowest risk-adjusted returns upon recovery; and (ii) the extent of belief differences about the recovery arrival rate. However, this expression cannot be directly used to determine a value for $\xi_{2\to 3}$, because risk-adjusted returns are unobservable. To make progress, we note, from (IA.3),

$$\frac{V_{a,t,s'}^i(a_{t,s'}^i)}{V_{a,t,s}^i(a_{t,s}^i)} = \frac{\alpha_{t,s}^i Q_{t,s}}{\alpha_{t,s'}^i Q_{t,s'}}.$$

In an economy with only agents with belief type i, we have $\alpha_{t,s}^i = \alpha_{t,s'}^i$ (i.e., upon transition, the economy continues to have only agents with belief type i). Therefore, in such economies,

$$\frac{V_{a,t,s'}^i(a_{t,s'}^i)}{V_{a,t,s}^i(a_{t,s}^i)} = \frac{Q_{t,s}}{Q_{t,s'}}. \tag{IA.31}$$

Combining Eqs. (IA.28) and (IA.31) with the definition of the net return $r_{s\to s'}^i$ in Section 2 for the economy in which all investors have belief type i, we obtain that

$$|r_{2\to 3}^{i,adj}| = |r_{0\to 2}^i|. \tag{IA.32}$$

That is, in the economy with only agents with belief i, the magnitude of the *risk-adjusted* return upon recovery is equal to the magnitude of the *non-risk-adjusted* return upon bust under Wicksellian policy. Substituting Eq. (IA.32) into (IA.30), we obtain Eq. (25), which is in terms of non-risk-adjusted returns (which are observable). Eq. (25) is obtained by subtracting the version of Eq. (IA.29) for $i = p$ from the version for $i = o$. If instead one adds these versions of Eq. (IA.29), we obtain an expression for g_2 such that, given our externally set parameter values, Eq. (26) is an accurate approximation.

Similarly, we can use Eq. (IA.29) for $(s, s') = (0, 2)$ to obtain an expression akin to Eq. (IA.30) for $\xi_{0\to 2}$ in terms of $r_{0\to 2}^{i,adj}$. The steps are the same, except that, in the boom state $s = 0$, $r_{t,s}^f > 0$ and hence the expression for $\xi_{0\to 2}$ also contains $r_0^{f,i}$, the risk-free rate in state 0 when all investors have beliefs i. (These risk-free rate terms do not appear in Eq.

[9]The approximation $\delta(\eta_2^o) \approx \delta(\eta_2^p)$ holds very well in our calibrated model and when using the parameter values in Caballero and Simsek (2020a). The approximation holds exactly in the $\delta(\eta) = \bar{\delta}$ case. Note that Eq. (IA.29) always holds with equality in $s = 2$, for $i \in \{o, p\}$.

(IA.29) with $s = 2$ because $r^f_{t,2} = 0$.) Moreover, similar to Eq. (IA.32), we have

$$|r^{i,adj}_{0\to 2}| = |r^i_{2\to 3}|.$$

Combining these results, one obtains Eq. (27) of the main text.[10]

C.2 Rate-of-return data and leverage parameter

For calculating the lengths of business cycles, we use NBER data (National Bureau of Economic Research 2026) for 1952–2023. For data on the real risky return, we use the Jordà-Schularick-Taylor macrohistory database (Jordà et al. 2017, 2019, 2022). The real risky return is the nominal return on risky assets, *risky_tr*, less inflation measured using consumer prices, *cpi*. Return data are available through 2020.

The leverage parameter is connected to the sensitivity of intermediaries' capital to declines in risky asset prices upon a bust. We use data from He et al. (2017, 2026). We denote the capital ratio in state s by $\kappa_{s,t}$. In our model, upon transition to state s' at time t, the change in the capital ratio is related to the change in levered investors' wealth share according to $\frac{\kappa_{t,s'}}{\kappa_{t,s}} = \frac{\alpha_{t,s'}}{\alpha_{t,s}}$. The budget constraint also implies Eq. (13) of the main text: $\frac{\alpha_{t,s'}}{\alpha_{t,s}} = 1 + (\omega_{t,s} - 1)\left(1 - \frac{Q_{t,s}}{Q_{t,s'}}\right)$. Combining these results with the definition of the net return upon state transition, $r_{s\to s'} = \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}$, we obtain

$$\left(\frac{\kappa_{t,s'}}{\kappa_{t,s}} - 1\right)\frac{1 + r_{s\to s'}}{r_{s\to s'}} + 1 = \omega^o_{t,s}. \tag{IA.33}$$

Using Eq. (IA.33), we take an average across the starts of recessions with a negative real risky return to obtain the $\bar{\omega}$ reported in Table 1. As an example, in the 1973 recession, the intermediary capital *ratio* fell by 19% ($\frac{\kappa_{t,s'}}{\kappa_{t,s}} - 1 = -0.19$). This is equivalent to a decline in intermediary capital (or equity) of 24% ($(\frac{\kappa_{t,s'}}{\kappa_{t,s}})(1 + r_{s\to s'}) - 1 = -0.24$). With a risky return of -6%, the example of the 1973 recession is consistent with leverage $\bar{\omega}$ equal to approximately $\frac{24\%}{6\%}$ or 4. Sensitivity analyses presented in Section 5.4 and Appendix E.4 show key quantitative results for a range of alternative values for the leverage ratio.

Alternative calibration based on the GFC. Our benchmark calibration focuses on U.S. recessions. As an alternative, we use cross-country return data for the Global Financial

[10] The definition of $\xi_{s\to s'}$ in Eq. (23) uses an absolute value, but in Section 2 we omit $|\cdot|$ in Eq. (27) because it is redundant under our parametric restrictions. That is, we label types so that $\lambda^p_{0\to 2} > \lambda^o_{0\to 2}$ and $\lambda^p_{2\to 3} < \lambda^o_{2\to 3}$. We also obtain $0 < r^o_{2\to 3} < r^p_{2\to 3}$ and $r^{f,o}_0 > r^{f,p}_0 > 0$. Reinstating absolute values would not change any calibrated values. A similar argument applies to Eqs. (24) and (25).

Table IA.I: **Parameter values in the alternative calibration**

Parameter	Description	Value
g_2	Growth rate of capital during recession	-0.04
$(\lambda^o_{s\to 2}, \quad \lambda^{pl}_{s\to 2}, \quad \lambda^p_{s\to 2})$	Transition rate from boom ($s \in \{0,1\}$) to recession	(0.08, 0.20, 0.32)
$(\lambda^o_{2\to 3}, \quad \lambda^{pl}_{2\to 3}, \quad \lambda^p_{2\to 3})$	Transition rate from recession $s=2$ to recovery $s=3$	(1.79, 1.00, 0.21)
$\bar{\omega}$	Leverage limit	4.4

Crisis (GFC). As proxies for the return upon recession in the all-pessimist and all-optimist economies, we use, respectively, the lowest (-18%) and highest (-2%) real return on risky assets in 2008 associated with the GFC, across the 18 countries in the Jordà et al. (2022) data.[11] Similarly, to proxy for the return upon recovery in the all-pessimist and all-optimist economies, we use the highest and lowest real return on risky assets in 2010. We also condition on a range of variation of 5.4% for the real risk-free policy rate during the boom ($r_0^{f,o} - r_0^{f,p} = 5.4\%$), based on the real federal funds rate in the United States in the boom preceding the GFC; the data are from Board of Governors of the Federal Reserve System (US) and US Bureau of Economic Analysis (2026). In the U.S., intermediaries' capital ratio (He et al. 2017) fell by about 70% from the end of 2006 until the end of 2008, from approximately 0.08 to 0.03, amid a real return on risky assets from Jordà et al. (2019) over 2007 and 2008, equal to $1.029 \cdot 0.825 - 1 = -15.1\%$. We retain the same parameter values for the discount rate ρ and the depreciation parameters as in our benchmark. Parameter values that differ in the GFC-based, cross-country calibration are shown in Table IA.I. Figure IA.4 shows a version of main-text Figure 5 using our alternative calibration.

C.3 Additional numerical results

Forward rate curve, with and without state transitions. Figure IA.5 shows the forward rate curve in $s=1$, absent state transition (left panel of Figure IA.5, from Eq. (39), and as also shown in the bottom-right panel of Figure 5 in the main text) and taking into account the possibility of state transition (right panel of Figure IA.5). The forward expected policy rate (at time τ) over m years, beginning h years ahead, taking into account the possibility

[11] In some countries, the recession did not begin until 2009; adjusting the calculation to take this into account, the results are very similar.

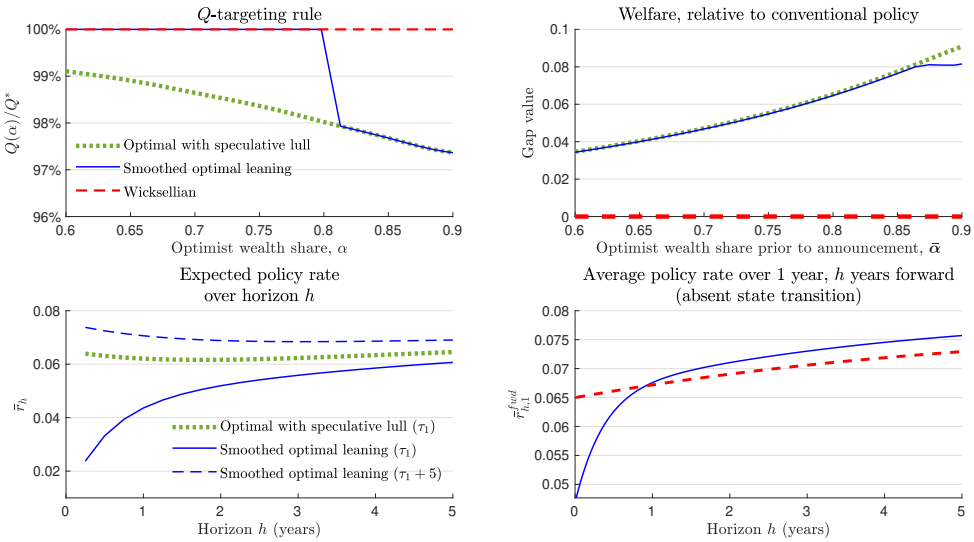


Figure IA.4: **Optimal leaning, under the alternative calibration**.

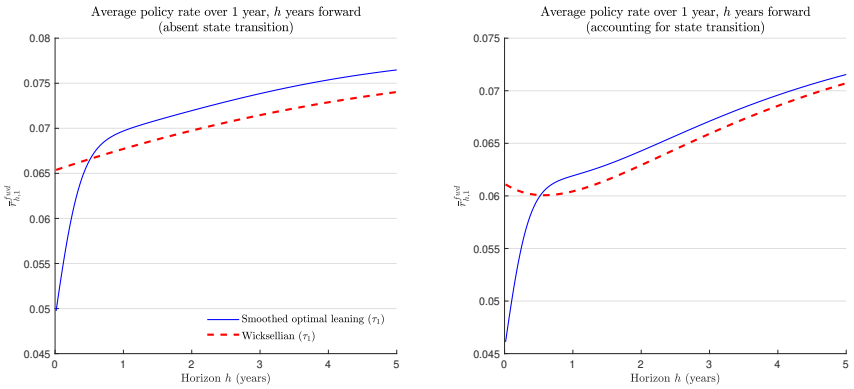


Figure IA.5: **Forward rate curves, with and without accounting for state transitions**

of state transitions, is $\frac{1}{m}\mathbb{E}^i_\tau[\int_{\tau+h}^{\tau+h+m} r^f_{t,s}dt]$. Accounting for state transitions, it remains the case that forward rates under smoothed optimal policy exceed those under Wicksellian policy, beyond an initial period, consistent with the results when conditioning on the absence of state transition.

Relevant classes of leaning rules. The leaning rule Eq. (28) has similar properties to the Q-targeting rule for the prudential monetary policies (PMP) in Caballero and Simsek (2020a) that mimic the macroprudential benefits of tightening regulation (i.e., leverage limit $\tilde{\omega} < \bar{\omega}$). This PMP rule calls for $Q_1(\alpha) = Q^*$ whenever the leverage limit does not

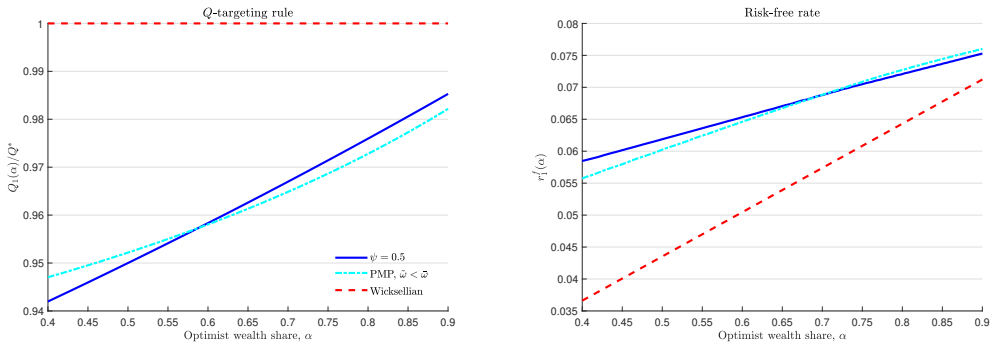


Figure IA.6: **Comparing relevant classes of leaning rules**.

bind in the economy with leverage limit $\tilde{\omega}$. Otherwise, $Q_1(\alpha)$ is set according to

$$Q_1(\alpha) = Q_2(\alpha_{1\to 2}(\alpha;\tilde{\omega}))\left(1+\frac{\tilde{\omega}-1}{\bar{\omega}-1}\left(\frac{Q^*}{Q_2(\alpha_{1\to 2}(\alpha;\tilde{\omega}))}-1\right)\right).$$

Figure IA.6 shows that the leaning rule Eq. (28) has similarities with this PMP rule.

D. Proofs and derivations: Announcement effect

This appendix provides proofs and derivations pertaining to Sections 3–4 of the main text.

Q-targeting rule in $s=1$. We restrict attention to Q-targeting rules in $s=1$ satisfying $\frac{Q_1'(\alpha)}{Q_1(\alpha)} < 1$. That is, the elasticity of the asset price to optimists' wealth share is not too high. This is a sufficient condition that can be relaxed.

Proof of Proposition 1. We first sketch the proof. If $Q_1(\bar{\boldsymbol{\alpha}}) = Q^*$, then $\alpha_{0\to 1}(\bar{\boldsymbol{\alpha}}) = \bar{\boldsymbol{\alpha}}$ is a solution to Eq. (29). For the case $Q_1(\bar{\boldsymbol{\alpha}}) < Q^*$, a heuristic version of the proof can be seen immediately from Figure IA.7. Optimists' wealth share after announcement (conditional on asset price Q_1 and shown in purple) is increasing in Q_1 and includes the point $(\bar{\boldsymbol{\alpha}}, Q^*)$. Hence, a Q-target with $Q_1(\bar{\boldsymbol{\alpha}}) < Q^*$ implies $\alpha_{0\to 1}(\bar{\boldsymbol{\alpha}}) < \bar{\boldsymbol{\alpha}}$.

Next, we proceed to the complete proof of the existence and uniqueness of $\alpha_{0\to 1}(\alpha)$ and the properties claimed in Proposition 1. Optimist leverage in the $s=0$ boom is $\omega_0^o(\alpha) = \min\{\bar{\omega}, \omega_0^{unc}(\alpha)\} \in (1, \bar{\omega}]$, where the optimist leverage, if unconstrained in boom state $s \in \{0,1\}$, is

$$\omega_s^{unc}(\alpha) = 1 + \frac{\frac{\lambda_{s\to 2}^o}{\lambda_{s\to 2}(\alpha)} - 1}{1 - \frac{Q_s(\alpha)}{Q_2\left(\frac{\lambda_{s\to 2}^o}{\lambda_{s\to 2}(\alpha)}\alpha\right)}} > 1 \tag{IA.34}$$

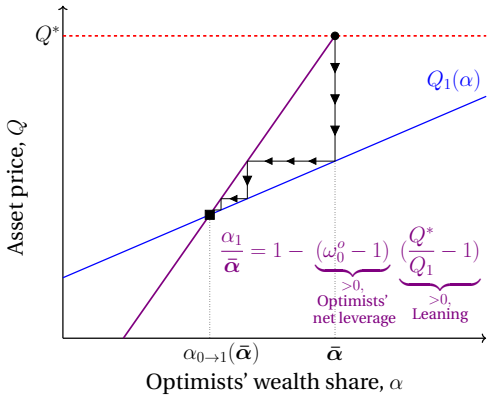


Figure IA.7: **The announcement effect**. In $s = 1$, if the central bank maintains conventional policy, $Q_1(\alpha) = Q^*$, as shown by the red dashed horizontal line. Leaning, instead, reduces the risky asset price according to the Q-targeting rule $Q_1(\alpha) < Q^*$, as shown by the solid blue line. Just prior to the adoption of leaning, optimists' wealth share is $\bar{\boldsymbol{\alpha}}$ and optimists have gross leverage $\omega_0^o > 1$. The new, lower equilibrium optimist wealth share and risky asset price are shown by the black square.

Using this result ($\omega_0^o(\alpha) \in (1, \bar{\omega}]$) and Eq. (29), $Q_1(\bar{\boldsymbol{\alpha}}) = Q^*$ implies $\alpha_{0\to1}(\bar{\boldsymbol{\alpha}}) = \bar{\boldsymbol{\alpha}}$. The remainder of the proof focuses on the case $Q_1(\bar{\boldsymbol{\alpha}}) < Q^*$. For $(\alpha_1, \alpha_0) \in (0,1) \times (0,1)$, define the function $f(\alpha_1, \alpha_0)$ as

$$f(\alpha_1, \alpha_0) \equiv \alpha_0 \left(1 - (\omega_0^o(\alpha_0) - 1)\left(\frac{Q^*}{Q_1(\alpha_1)} - 1\right)\right). \tag{IA.35}$$

Optimists' wealth share following adoption, $\alpha_{0\to1}(\bar{\boldsymbol{\alpha}})$, satisfies the fixed-point condition $f(\alpha_{0\to1}(\bar{\boldsymbol{\alpha}}); \bar{\boldsymbol{\alpha}}) = \alpha_{0\to1}(\bar{\boldsymbol{\alpha}})$, as can be seen by comparing Eqs. (29) and (IA.35). To complete the proof, we will show that (i) $f(\alpha_{0\to2}(\alpha); \alpha) \geq \alpha_{0\to2}(\alpha)$ for all $\alpha \in (0,1)$; (ii) $f(\bar{\boldsymbol{\alpha}}; \bar{\boldsymbol{\alpha}}) < \bar{\boldsymbol{\alpha}}$; and (iii) if $\alpha_1 \in (0,1)$ satisfies $f(\alpha_1; \bar{\boldsymbol{\alpha}}) = \alpha_1$, it holds that $f_{\alpha_1}(\alpha_1, \bar{\boldsymbol{\alpha}}) < 1$.

Starting with claim (i): The optimist wealth share $\alpha_{0\to2}(\alpha)$ that obtains if the economy transitions from $s = 0$ directly into a recession satisfies the fixed-point condition:

$$\alpha_{0\to2}(\alpha) = \alpha\left(1 - (\omega_0^o(\alpha) - 1)\left(\frac{Q^*}{Q_2(\alpha_{0\to2}(\alpha))} - 1\right)\right), \tag{IA.36}$$

where α is the optimist wealth share just prior to recession. Combining Eqs. (IA.35), (IA.36), and $Q_1(\alpha) \geq Q_2(\alpha)$, one obtains $f(\alpha_{0\to2}(\alpha), \alpha) \geq \alpha_{0\to2}(\alpha)$, for $\alpha \in (0,1)$. Thus, we have condition (i).

Condition (ii) follows immediately from $Q_1(\bar{\boldsymbol{\alpha}}) < Q^*$ and $\omega_0^o(\bar{\boldsymbol{\alpha}}) > 1$.

For condition (iii), differentiate Eq. (IA.35) to obtain

$$f_{\alpha_1}(\alpha_1, \alpha_0) = \alpha_0(\omega_0^o(\alpha_0) - 1)\frac{Q^*}{Q_1(\alpha_1)^2}Q_1'(\alpha_1).$$

This result and $f(\alpha_1, \alpha_0) = \alpha_1$ imply

$$f_{\alpha_1}(\alpha_1, \alpha_0) = (\alpha_0 \omega_0^o(\alpha_0) - \alpha_1) \frac{Q_1'(\alpha_1)}{Q_1(\alpha_1)}. \tag{IA.37}$$

Using again $f(\alpha_1, \alpha_0) = \alpha_1$, we have

$$\alpha_0 \omega_0^o(\alpha_0) - \alpha_1 = (\alpha_0 - \alpha_1) \left(\frac{Q^*}{Q^* - Q_1(\alpha_1)} \right). \tag{IA.38}$$

Next, note that $\bar{\omega} < \hat{\omega}$ implies $\omega_0^o(\alpha_0) \leq \bar{\omega} < 1 + \frac{Q_2(0)}{Q^*}$, because $Q_2(0) < Q^*$. In addition, $\bar{\omega} < 1 + \frac{Q_2(0)}{Q^*}$ implies $\alpha - \alpha_{0\to1}(\alpha) < 1 - \frac{Q_1(\alpha_{0\to1}(\alpha))}{Q^*}$ for any $\alpha \in (0,1)$. Combining this result with Eqs. (IA.37)- (IA.38), we obtain $f_{\alpha_1}(\alpha_1, \bar{\boldsymbol{\alpha}}) < 1$ if $f(\alpha_1, \bar{\boldsymbol{\alpha}}) = \alpha_1$ (claim (iii)).

Proof of Proposition 2. In the proofs, we refer to Q-targeting rules of the form Eq. (35) as "ramp" rules because such rules phase in or "ramp up" leaning over time. We denote these rules by $Q_1^{\text{ramp}}(\alpha; \bar{\boldsymbol{\alpha}}, \hat{\alpha})$, with this notation explicitly denoting the dependence on the wealth share prior to the policy change $\bar{\boldsymbol{\alpha}}$ and the adjustment parameter $\hat{\alpha}$.

Proof of part (i). Assume that $Q^{\text{lull}}(\bar{\boldsymbol{\alpha}}) = Q^*$. Define the admissible rule

$$Q_1^{\text{opt}}(\alpha; \bar{\boldsymbol{\alpha}}) \equiv \begin{cases} Q^*, & \alpha < \bar{\boldsymbol{\alpha}}, \\ Q^{\text{lull}}(\alpha), & \alpha \geq \bar{\boldsymbol{\alpha}}. \end{cases}$$

Because $Q^{\text{lull}}(\bar{\boldsymbol{\alpha}}) = Q^*$, the rule $Q_1^{\text{opt}}(\cdot; \bar{\boldsymbol{\alpha}})$ is continuous at $\alpha = \bar{\boldsymbol{\alpha}}$ and right-differentiable (hence in $\mathcal{C}_r$). Moreover, $Q_1^{\text{opt}}(\bar{\boldsymbol{\alpha}}; \bar{\boldsymbol{\alpha}}) = Q^*$, so Proposition 1(ii) implies $\alpha_{0\to1}(\bar{\boldsymbol{\alpha}}) = \bar{\boldsymbol{\alpha}}$ under $Q_1^{\text{opt}}(\cdot; \bar{\boldsymbol{\alpha}})$. Therefore there is no decline in optimists' wealth share upon announcement (setting $Q_1(\alpha) = Q^*$ for $\alpha \leq \bar{\boldsymbol{\alpha}}$ rules out an equilibrium with $\alpha_{0\to1}(\bar{\boldsymbol{\alpha}}) < \bar{\boldsymbol{\alpha}}$). For all $\alpha \geq \bar{\boldsymbol{\alpha}}$ the rule coincides with the speculative-lull optimum $Q^{\text{lull}}(\alpha)$, delivering gap value $w_1^{\text{lull}}(\bar{\boldsymbol{\alpha}})$.

Proof of part (ii). For any $\hat{\alpha}$ and $\bar{\boldsymbol{\alpha}}$, the rules $Q^{\text{lull}}(\alpha)$ and $Q_1^{\text{ramp}}(\alpha; \bar{\boldsymbol{\alpha}}, \hat{\alpha})$ coincide (as functions of α) for $\alpha \in [\bar{\boldsymbol{\alpha}} + \hat{\alpha}, 1]$. In contrast, by construction, the rules may differ over $[\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\alpha}} + \hat{\alpha})$.

Shifting to the time domain and integrating forward the ODE for $w_{t,1}^{pl}$, we have:

$$w_{t,1}^{pl} = \int_0^\infty e^{-(\rho + \lambda_{1\to2}^{pl})\vartheta} \left[W(Q_{t+\vartheta,1}) + \lambda_{1\to2}^{pl} w_{t+\vartheta,2}^{pl} \right] d\vartheta.$$

Fix the initial condition, $\alpha_{\tau,1} = \bar{\boldsymbol{\alpha}}$ (i.e., let τ be the arrival time of $s = 1$).[12] We use

[12] Under the ramp rule (without a lull) and under the lull rule (with a lull), the optimist wealth share does not change upon transition to $s = 1$. Hereafter in this proof, for notational convenience, we drop the pl

$j \in \{\text{ramp}, \text{lull}\}$ to index the two Q-targeting rules. Under rule j, define the hitting time for $\alpha^j_{t,1}$ reaching $\bar{\boldsymbol{\alpha}} + \hat{\alpha}$ as

$$\tau_j^{hit}(\hat{\alpha}) \equiv \inf\{t \geq \tau : \alpha^j_{t,1} \geq \bar{\boldsymbol{\alpha}} + \hat{\alpha}\}.$$

Since $\dot{\alpha}_1^j(\alpha)$ is continuous in α and strictly positive for $\alpha \in (0,1)$ (for each $j \in \{\text{ramp}, \text{lull}\}$), choose $\delta \in (0, 1 - \bar{\boldsymbol{\alpha}})$ such that $[\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\alpha}} + \delta] \subset (0,1)$ and define

$$m \;\equiv\; \min_{j \in \{\text{ramp},\text{lull}\}} \; \min_{\alpha \in [\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\alpha}} + \delta]} \dot{\alpha}_1^j(\alpha) \;>\; 0.$$

Hence, for any $\hat{\alpha} \in (0, \delta]$ and each $j \in \{\text{ramp}, \text{lull}\}$,

$$\tau_j^{hit}(\hat{\alpha}) - \tau = \int_{\bar{\boldsymbol{\alpha}}}^{\bar{\boldsymbol{\alpha}} + \hat{\alpha}} \frac{1}{\dot{\alpha}_1^j(u)}\, du \;\leq\; \frac{\hat{\alpha}}{m}.$$

Therefore,

$$t^*(\hat{\alpha}) \;\equiv\; \max_j \left(\tau_j^{hit}(\hat{\alpha}) - \tau\right) \;\leq\; \frac{\hat{\alpha}}{m} \;=\; \mathcal{O}(\hat{\alpha}),$$

and in particular $t^*(\hat{\alpha}) \to 0$ as $\hat{\alpha} \to 0^+$.

We then have

$$|w^{\text{ramp}}_{\tau,1} - w^{\text{lull}}_{\tau,1}| \leq \underbrace{\int_0^{t^*(\hat{\alpha})} e^{-(\rho + \lambda^{pl}_{1\to 2})\vartheta} \left[|\Delta W(Q_{\tau+\vartheta,1}) + \lambda^{pl}_{1\to 2} \Delta w_{\tau+\vartheta,2}|\right] d\vartheta}_{\psi_1} +$$
$$\underbrace{\int_{t^*(\hat{\alpha})}^{\infty} e^{-(\rho + \lambda^{pl}_{1\to 2})\vartheta} \left[|\Delta W(Q_{\tau+\vartheta,1}) + \lambda^{pl}_{1\to 2} \Delta w_{\tau+\vartheta,2}|\right] d\vartheta}_{\psi_2},$$

where $\Delta W(Q_{\tau+\vartheta,1}) = W(Q^{\text{ramp}}_{\tau+\vartheta,1}) - W(Q^{\text{lull}}_{\tau+\vartheta,1})$ and $\Delta w_{\tau+\vartheta,2}$ is defined analogously.

For the ψ_1 term, the horizon length is $\mathcal{O}(\hat{\alpha})$. Observe that $W(Q)$ is continuous in Q and $w_2(\alpha)$ and $\alpha_{1\to 2}(\alpha)$ are each continuous in α. Therefore, the integrand (in ψ_1) is bounded. Hence, $\psi_1 = \mathcal{O}(\hat{\alpha})$.

For the ψ_2 term, we show that $\lim_{\hat{\alpha}\to 0} \psi_2 = 0$, based on continuous dependence and dominated convergence. For brevity, we drop the explicit dependence of $t^*(\hat{\alpha})$ on $\hat{\alpha}$ in the remainder of the proof. Note that $\lim_{\hat{\alpha}\to 0} |\alpha^{\text{ramp}}_{\tau+t^*,1} - \alpha^{\text{lull}}_{\tau+t^*,1}| = 0$. Observe also that $\lim_{t\to\infty} \alpha^j_{t,1} = 1$ for $j \in \{\text{ramp}, \text{lull}\}$. For any $\alpha \in [\bar{\boldsymbol{\alpha}}, 1]$, we also have that $Q_1^j(\alpha) \in [Q_2^p, Q^*]$, where Q_2^p is the asset price in the all-pessimist $s = 2$ economy. Therefore $W(Q_1^j(\alpha))$ and $w_2^j(\alpha_{1\to 2}(\alpha; Q_1^j(\alpha)))$ are also bounded on $\alpha \in [\bar{\boldsymbol{\alpha}}, 1]$.

superscript that indicates that gap values are calculated under the planner's beliefs.

For the ψ_2 term, over the integration interval, $Q_1^{\text{ramp}}(\alpha) = Q^{\text{lull}}(\alpha)$ and hence $\dot{\alpha}_1^{\text{ramp}}(\alpha) = \dot{\alpha}_1^{\text{lull}}(\alpha)$. That is, the Q-targeting rules and growth rates of α coincide (as functions of α). Therefore, the paths for $\alpha_{t,1}^{\text{ramp}}$ and $\alpha_{t,1}^{\text{lull}}$ differ over the integration interval only because $\alpha_{\tau+t^*,1}^{\text{ramp}}$ may not equal $\alpha_{\tau+t^*,1}^{\text{lull}}$. From $\tau + t^*$ onward, the optimist wealth share under each rule evolves under the same ODE $\dot{\alpha}_1^{\text{lull}}(\alpha)$, which is locally Lipschitz on $[\bar{\boldsymbol{\alpha}}, 1]$. Under either rule, for $\vartheta \in [t^*, T]$ with $T < \infty$, we have $\alpha_{\tau+\vartheta,1}^j \in [\alpha_{\tau+t^*,1}^j, \alpha_{\tau+T,1}^j]$, a compact set.

Taking these observations together and using the Cauchy–Lipschitz theorem (and Grönwall's inequality), we obtain continuous dependence: for any $T > t^*(\hat{\alpha})$,

$$\lim_{\hat{\alpha}\to 0} \sup_{\vartheta\in[t^*,T]} |\alpha_{\tau+\vartheta,1}^{\text{ramp}} - \alpha_{\tau+\vartheta,1}^{\text{lull}}| = 0.$$

By the continuity of $W(Q)$ and $w_2(\alpha_{1\to 2}(\alpha; Q_1(\alpha)))$, we have that $W(Q)$ is bounded on $[Q_2^p, Q^*]$ and $w_2(\alpha_{1\to 2}(\alpha; Q_1(\alpha)))$ is bounded on $[\bar{\boldsymbol{\alpha}}, 1]$. Define

$$\Delta_{\hat{\alpha}}(\vartheta) = \mathbb{1}_{\vartheta > t^*(\hat{\alpha})} e^{-(\rho+\lambda_{1\to 2}^{pl})\vartheta} \left[\Delta W(Q_{\tau+\vartheta,1}) + \lambda_{1\to 2}^{pl} \Delta w_{\tau+\vartheta,2}\right].$$

Then, for

$$M = \sup_{\substack{Q\in[Q_2^p,Q^*]\\ \alpha\in[\bar{\boldsymbol{\alpha}},1]}} \left(|W(Q)| + \lambda_{1\to 2}^{pl} \left|w_2(\alpha_{1\to 2}(\alpha; Q))\right|\right) < \infty,$$

we have

$$|\Delta_{\hat{\alpha}}(\vartheta)| \le 2Me^{-(\rho+\lambda_{1\to 2}^{pl})\vartheta},$$

where the right side of the inequality is integrable. For any $\vartheta > 0$, observe that $\mathbb{1}_{\vartheta > t^*(\hat{\alpha})} \to 1$ and $[\Delta W(Q_{\tau+\vartheta,1}) + \lambda_{1\to 2}^{pl} \Delta w_{\tau+\vartheta,2}] \to 0$ as $\hat{\alpha} \to 0$. By the dominated convergence theorem, we therefore have $\lim_{\hat{\alpha}\to 0} \psi_2 = 0$.

Proof of Proposition 3. Fix a horizon $h > 0$. Consider the expected policy rate at time t in state s over horizon h (taking into account the potential for state transitions), defined in Eq. (37), under the planner's beliefs, $\bar{r}_{h,t,s}^{pl}$. Denote the arrival time of state s by τ_s. Denote by $\bar{r}^{\text{lull}}$ the expected policy rate $\bar{r}_{h,\tau_1,1}^{pl}$ when the Q-target $Q^{\text{lull}}(\alpha)$ is announced during a lull. Denote by $\bar{r}^{\text{ramp}}(\hat{\alpha})$ the expected policy rate $\bar{r}_{h,\tau_1,1}^{pl}$ when the Q-target $Q_1^{\text{ramp}}(\alpha; \bar{\boldsymbol{\alpha}}, \hat{\alpha})$ is announced. Define $\mathcal{T} = \min\{\tau_2 - \tau_1, h\}$.

$$\bar{r}^{\text{ramp}}(\hat{\alpha}) - \bar{r}^{\text{lull}} = \frac{1}{h}\mathbb{E}_{\tau_1}^{pl}\left[\int_0^{\mathcal{T}} \left(r_1^{f,\text{ramp}}(\alpha_{\tau_1+\vartheta}^{\text{ramp}}) - r_1^{f,\text{lull}}(\alpha_{\tau_1+\vartheta}^{\text{lull}})\right) d\vartheta\right].$$

There is no contribution to $\bar{r}^{\text{ramp}}(\hat{\alpha}) - \bar{r}^{\text{lull}}$ from $s \in \{2, 3\}$, because the risk-free rate

in these states is policy invariant (at time t, the risk-free rate is $r^f_{t,2} = 0$ if $s = 2$ and $r^f_{t,3} = \rho + g_3 - \delta(\eta^*) > 0$ if $s = 3$).

The implementing risk-free rate for Q-targeting rule $Q_1(\alpha)$ is given by Eq. (21), where $\alpha_{1\to 2}(\alpha)$ and $\dot\alpha_1(\alpha)$ are given by Eqs. (13) and (15). For a given rule $Q_1(\alpha)$, consider the expression, $\frac{d\ln(Q_1(\alpha))}{d\alpha}\dot\alpha_1(\alpha)$, which is a term in the implementing rule, Eq. (21). Under this Q-targeting rule, we have[13]

$$\bar r^{pl}_{h,\tau_1,1} \equiv \frac{1}{h}\mathbb{E}^{pl}_{\tau_1}\left[\int_0^{\mathcal{T}}\left(\frac{d\ln(Q_1(\alpha))}{d_+\alpha}\dot\alpha_1(\alpha)\right)d\vartheta\right] + \frac{1}{h}\mathbb{E}^{pl}_{\tau_1}\left[\int_0^{\mathcal{T}}\left(\rho + g_1 - \delta\left(\frac{Q_1(\alpha)}{Q^*}\eta^*\right) - \frac{(1-\alpha)\lambda^p_{1\to 2}}{1-\alpha_{1\to 2}(\alpha)}\left(\frac{Q_1(\alpha)}{Q_2(\alpha_{1\to 2}(\alpha))} - 1\right)\right)d\vartheta\right] + C^{pl}_h, \tag{IA.39}$$

where C^{pl}_h collects terms that are policy invariant. Examining the top line of Eq. (IA.39), we have, for given stopping time $\mathcal{T}$,

$$\int_0^{\mathcal{T}}\left(\frac{d\ln(Q_1(\alpha))}{d_+\alpha}\dot\alpha_1(\alpha)\right)d\vartheta = \int_0^{\mathcal{T}}\frac{d}{d\vartheta}\ln(Q_1(\alpha))d\vartheta = \ln Q_1(\alpha_{\tau_1+\mathcal{T}}) - \ln Q_1(\alpha_{\tau_1}). \tag{IA.40}$$

Next, consider the difference between (IA.40) under the rules, $j \in \{\text{ramp}, \text{lull}\}$.

$$\int_0^{\mathcal{T}}\left(\frac{d\ln(Q_1^{\text{ramp}}(\alpha))}{d_+\alpha}\dot\alpha_1^{\text{ramp}}(\alpha) - \frac{d\ln(Q^{\text{lull}}(\alpha))}{d_+\alpha}\dot\alpha_1^{\text{lull}}(\alpha)\right)d\vartheta = \underbrace{\left(\ln Q_1^{\text{ramp}}(\alpha^{\text{ramp}}_{\tau_1+\mathcal{T}}) - \ln Q^{\text{lull}}(\alpha^{\text{lull}}_{\tau_1+\mathcal{T}})\right)}_{\varsigma_1(\hat\alpha)} - \underbrace{\left(\ln Q_1^{\text{ramp}}(\bar{\boldsymbol{\alpha}}) - \ln Q^{\text{lull}}(\bar{\boldsymbol{\alpha}})\right)}_{\varsigma_2}.$$

Using the definition of $Q_1^{\text{ramp}}(\alpha)$, we have $\varsigma_2 = \ln\left(\frac{Q^*}{Q^{\text{lull}}(\bar{\boldsymbol{\alpha}})}\right)$.

Next, we show that $\mathbb{E}^{pl}_{\tau_1}[\varsigma_1(\hat\alpha)] \to 0$ as $\hat\alpha \to 0^+$. Decompose

$$\mathbb{E}^{pl}_{\tau_1}[\varsigma_1(\hat\alpha)] = \mathbb{E}^{pl}_{\tau_1}[\varsigma_1(\hat\alpha)\mathbf{1}\{\mathcal{T} \le t^*(\hat\alpha)\}] + \mathbb{E}^{pl}_{\tau_1}[\varsigma_1(\hat\alpha)\mathbf{1}\{\mathcal{T} > t^*(\hat\alpha)\}].$$

On $\{\mathcal{T} \le t^*(\hat\alpha)\}$ we have

$$Q_1^{\text{ramp}}\left(\alpha^{\text{ramp}}_{\tau_1+\mathcal{T}}\right),\ Q^{\text{lull}}\left(\alpha^{\text{lull}}_{\tau_1+\mathcal{T}}\right) \in [Q^p_2, Q^*],$$

[13] To ease notation, in Eq. (IA.39) and the subsequent two equations, whenever α appears in the integrand, the time subscript $\tau_1 + \vartheta$ is omitted. Note that, although $\bar r^{pl}_{h,\tau_1,1}$ is defined using the planner's expectation operator $\mathbb{E}^{pl}_{\tau_1}$, the integrand in (IA.39) is the implementing policy rate from (21). Hence the recession-risk term continues to use the pessimists' bust-arrival belief, $\lambda^p_{1\to 2}$; the planner's beliefs affect $\bar r^{pl}_{h,\tau_1,1}$ only through the distribution of the stopping time $\mathcal{T}$ (and the associated path of α) inside the expectation. Here, $C^{pl}_h \equiv \frac{1}{h}\mathbb{E}^{pl}_{\tau_1}\left[\int_{\mathcal{T}}^h r^f_{\tau_1+\vartheta,\tilde s_{\tau_1+\vartheta}}d\vartheta\right]$ collects the contribution from states $s = 2$ and $s = 3$. It is invariant to the $s = 1$ Q-targeting rule because, in those states, $r^f_{t,2} = 0$ and $r^f_{t,3} = \rho + g_3 - \delta(\eta^*)$, and transition rates are exogenous.

and hence

$$\left|\varsigma_1(\hat{\alpha})\right| = \left|\ln Q_1^{\text{ramp}}\left(\alpha_{\tau_1+\mathcal{T}}^{\text{ramp}}\right) - \ln Q^{\text{lull}}\left(\alpha_{\tau_1+\mathcal{T}}^{\text{lull}}\right)\right| \leq \ln\left(\frac{Q^*}{Q_2^p}\right) \equiv C.$$

Therefore,

$$\left|\mathbb{E}_{\tau_1}^{pl}[\varsigma_1(\hat{\alpha})\mathbf{1}\{\mathcal{T} \leq t^*(\hat{\alpha})\}]\right| \leq C \cdot \mathbb{P}_{\tau_1}^{pl}(\tau_2 - \tau_1 \leq t^*(\hat{\alpha})) = C\left(1 - e^{-\lambda_{1\to 2}^{pl} t^*(\hat{\alpha})}\right) \xrightarrow[\hat{\alpha}\to 0]{} 0,$$

because $t^*(\hat{\alpha}) \to 0$. We next address the $\{\mathcal{T} > t^*(\hat{\alpha})\}$ term. Over $\vartheta \in [0, t^*(\hat{\alpha})]$, the drift is bounded: there exists a $B > 0$ such that, for $\vartheta \in [0, t^*(\hat{\alpha})]$ and $j \in \{\text{ramp}, \text{lull}\}$, we have $|\dot{\alpha}_{\tau_1+\vartheta}^j| \leq B$. Therefore,

$$|\alpha_{\tau_1+t^*(\hat{\alpha})}^{\text{ramp}} - \alpha_{\tau_1+t^*(\hat{\alpha})}^{\text{lull}}| \leq \int_0^{t^*(\hat{\alpha})} \left(|\dot{\alpha}_1^{\text{ramp}}(\alpha_{\tau_1+\vartheta}^{\text{ramp}})| + |\dot{\alpha}_1^{\text{lull}}(\alpha_{\tau_1+\vartheta}^{\text{lull}})|\right) d\vartheta \leq 2Bt^*(\hat{\alpha}).$$

Observing that $2Bt^*(\hat{\alpha}) \to 0$ as $\hat{\alpha} \to 0^+$, we have that $|\alpha_{\tau_1+t^*(\hat{\alpha})}^{\text{ramp}} - \alpha_{\tau_1+t^*(\hat{\alpha})}^{\text{lull}}| \to 0$ as well. For $\vartheta \in [t^*(\hat{\alpha}), \mathcal{T}]$, the two Q-targeting rules (as functions of α) coincide and hence so do the growth rates of optimists' wealth share (as functions of α). Hence, by continuous dependence, and with $\ln Q_1(\cdot)$ continuous and right-differentiable over $[\bar{\boldsymbol{\alpha}}, 1]$, we have $\mathbb{E}_{\tau_1}^{pl}[\varsigma_1(\hat{\alpha})] \to 0$ as $\hat{\alpha} \to 0^+$.

Next, consider the second-line terms in (IA.39). On $[0, t^*(\hat{\alpha})]$, the integrand is bounded, so the contribution to the difference between the ramp and lull rules is at most $\tilde{C} \cdot t^*(\hat{\alpha}) = o(1)$ for some constant $\tilde{C} < \infty$. For $\vartheta \geq t^*(\hat{\alpha})$, the two targeting rules coincide as functions of α, so the second-line integrand can be written as a continuous function of α on a compact set. By uniform continuity and the fact that $\sup_{\vartheta \in [t^*(\hat{\alpha}), \mathcal{T}]} |\alpha_{\tau_1+\vartheta}^{\text{ramp}} - \alpha_{\tau_1+\vartheta}^{\text{lull}}| \to 0$, it follows that the difference in second-line contributions to (IA.39) vanishes as $\hat{\alpha} \to 0$.

Combining the previous steps, the first-line contribution in (IA.39) implies

$$\lim_{\hat{\alpha}\to 0^+} \left(\bar{r}^{\text{ramp}}(\hat{\alpha}) - \bar{r}^{\text{lull}}\right) = -\frac{1}{h}\varsigma_2 = -\frac{1}{h}\ln\left(\frac{Q^*}{Q^{\text{lull}}(\bar{\boldsymbol{\alpha}})}\right),$$

which yields (38). If $Q^{\text{lull}}(\bar{\boldsymbol{\alpha}}) < Q^*$, then the right-hand side is negative, so $\bar{r}_{h,\tau_1,1}^{\text{opt},i} < \bar{r}_{h,\tau_1,1}^{\text{lull},i}$ in the limit ("lower"). Differentiating (38) with respect to h gives $\frac{d\bar{r}_{h,\tau_1,1}^{opt,i}}{dh} = \frac{d\bar{r}_{h,\tau_1,1}^{\text{lull},i}}{dh} + \frac{1}{h^2}\ln\left(\frac{Q^*}{Q^{\text{lull}}(\bar{\boldsymbol{\alpha}})}\right)$, so the expected policy-rate curve is steeper in the limit.

Note that we write the proof under planner beliefs pl, for simplicity. For any $i \in \{o, p\}$, replace $\mathbb{E}^{pl}$ by $\mathbb{E}^i$ (and hence $\mathbb{P}^{pl}$ by $\mathbb{P}^i$, equivalently replacing the parameter $\lambda_{1\to 2}^{pl}$ governing the distribution of $\tau_2 - \tau_1$ by $\lambda_{1\to 2}^i$); the integrand in (IA.39) remains the implementing policy rate from (21) and therefore continues to use $\lambda_{1\to 2}^p$. The above argument is un-

changed. Moreover, the adjustment term in Eq. (38) is independent of i.

D.1 Additional results

The remainder of Appendix D provides additional results described in the main text of Section 3.

Lemma IA.3 *For a given optimists' wealth share α during the $s = 1$ boom, optimists' wealth share upon recession, $\alpha_{1\to 2}(\alpha)$, is increasing in α.*

Proof of Lemma IA.3. First, suppose that the leverage constraint is not binding: $\omega_1^{unc}(\alpha) < \bar{\omega}$, with $\omega_1^{unc}(\alpha)$ given by (IA.34). Then, from the top line of (IA.15), $\alpha_{1\to 2}(\alpha) = (1 + (\frac{1}{\alpha} - 1)\frac{\lambda_{1\to 2}^p}{\lambda_{1\to 2}^o})^{-1}$, which is increasing in α. Second, for α for which the leverage constraint is binding: Taking the derivative of the bottom line of (IA.15) with respect to α, we have

$$\frac{d\alpha_{s\to 2}(\alpha)}{d\alpha} = \frac{1}{1 - f_{\alpha_2}^{s\to 2}(\alpha_{s\to 2}(\alpha), \alpha)} \left(1 - (\bar{\omega} - 1)\left(\frac{Q_s(\alpha)}{Q_2(\alpha_{s\to 2}(\alpha))} - 1\right) - \alpha(\bar{\omega} - 1)\frac{Q_s'(\alpha)}{Q_2(\alpha_{s\to 2}(\alpha))}\right). \tag{IA.41}$$

Recall from claim (iii) in the Proof of Lemma IA.2 that $f_{\alpha_2}^{s\to 2}(\alpha_{s\to 2}(\alpha), \alpha) < 1$. Note that if $Q_s'(\alpha) \leq 0$, then expression (IA.41) is strictly positive, completing the proof for the case $Q_s'(\alpha) \leq 0$. We will next address the case of $Q_s'(\alpha) > 0$. Rearranging (IA.41), the condition $\frac{d\alpha_{s\to 2}(\alpha)}{d\alpha} > 0$ is equivalent to

$$\frac{\bar{\omega}}{\bar{\omega} - 1} > \frac{Q_s(\alpha)}{Q_2(\alpha_{s\to 2}(\alpha))}\left(1 + \alpha\frac{Q_s'(\alpha)}{Q_s(\alpha)}\right). \tag{IA.42}$$

Then, $Q_1(\alpha) \leq Q^*$ and $Q_2(0) \leq Q_2(\alpha_{1\to 2}(\alpha))$ and the assumptions $Q_1'(\alpha)/Q_1(\alpha) < 1$ and $\bar{\omega} < \hat{\omega}$ together imply (IA.42), completing the proof of the lemma.

Welfare implications of the announcement effect. Without a speculative lull, welfare upon announcing a leaning Q-targeting rule $Q_1(\cdot)$ is $w_1^{pl}(\alpha_{0\to 1}(\bar{\boldsymbol{\alpha}}))$. Welfare with a speculative lull is $w_1^{pl}(\bar{\boldsymbol{\alpha}})$. Denoting the welfare difference (due to the announcement effect) as $\Delta w_1^{pl}(\bar{\boldsymbol{\alpha}}) \equiv w_1^{pl}(\alpha_{0\to 1}(\bar{\boldsymbol{\alpha}})) - w_1^{pl}(\bar{\boldsymbol{\alpha}})$,

$$\Delta w_1^{pl}(\bar{\boldsymbol{\alpha}}) = \int_{\tau_1}^{\infty} e^{-(\rho+\lambda_{1\to 2}^{pl})(t-\tau_1)} \left[\underbrace{W(Q_1(\alpha_{t,1})) - W(Q_1(\alpha_{t,1}^{\text{lull}}))}_{\substack{\text{Welfare flow from different} \\ \text{utilization during boom at time } t}} + \lambda_{1\to 2}^{pl} \underbrace{(w_2^{pl}(\alpha_{1\to 2}(\alpha_{t,1})) - w_2^{pl}(\alpha_{1\to 2}(\alpha_{t,1}^{\text{lull}})))}_{\substack{\text{Welfare effect of different} \\ \text{expected utilization during recession}}}\right] dt,$$

where $\alpha_{t,1}^{\text{lull}}$ is the path for optimists' wealth share with a speculative lull, τ_1 is the time of transition to $s = 1$, and $\alpha_{t,1}$ continues to denote the optimist wealth share without a lull.

Proposition IA.2 (Announcement effect reduces welfare)

Consider a leaning monetary policy with Q-targeting rule satisfying $Q_1(\bar{\boldsymbol{\alpha}}) < Q^*$ and $Q_1(\alpha)$ weakly increasing in α. The announcement effect reduces welfare:

$$\Delta w_1^{pl}(\bar{\boldsymbol{\alpha}}) < 0.$$

Optimists' wealth share and utilization with the announcement effect are lower in the boom $(s = 1)$ and the recession $(s = 2)$ than with a speculative lull, $\forall\, t$ regardless of when the recession arises:

$$\alpha_{t,1} < \alpha_{t,1}^{\text{lull}} \qquad \textit{and} \qquad \alpha_{t,2} < \alpha_{t,2}^{\text{lull}}, \tag{IA.43}$$

$$\eta_{t,1} \leq \eta_{t,1}^{\text{lull}} \qquad \textit{and} \qquad \eta_{t,2} < \eta_{t,2}^{\text{lull}}. \tag{IA.44}$$

Discussion of Proposition IA.2. The announcement effect—the initial fall of optimists' wealth share when leaning monetary policy is adopted without a speculative lull—implies a lower path for the optimist wealth share throughout the boom and recession (expression (IA.43)), for each potential realization of the arrival times of recession and recovery. This result obtains because of three features of the equilibrium: (i) adopting leaning monetary policy without a speculative lull causes a fall, on impact, in optimists' wealth share; (ii) the law of motion for the optimist wealth share ($\dot{\alpha}_s(\alpha)$, $s \in \{1, 2, 3\}$) is identical whether or not a speculative lull occurred at the time of the leaning announcement; and (iii) optimists' wealth share upon recession is increasing in optimists' wealth share just prior to recession ($\alpha_{1\to 2}(\alpha)$ is increasing in α, per Lemma IA.3). The lower path for optimists' wealth share during the recession implies that utilization is strictly lower in the recession as a result of the announcement effect (second inequality in (IA.44)).

These results—the lower path for optimists' wealth share during boom and recession and lower utilization during the recession, when leaning is announced unexpectedly—do not depend on the assumption that the Q-targeting rule is weakly increasing in α. With the natural additional assumption that $Q_1(\alpha)$ is weakly increasing (implying that the risky asset price weakly appreciates over time during the boom), the lower path for optimists' wealth share during the boom also implies weakly lower utilization during the boom. Consequently, with the announcement effect reducing utilization during boom and recession, it follows that the announcement effect reduces welfare.

Proof of Proposition IA.2. From Proposition 1, $Q_1(\bar{\boldsymbol{\alpha}}) < Q^*$ implies $\alpha_{0\to1}(\bar{\boldsymbol{\alpha}}) < \bar{\boldsymbol{\alpha}}$. Thus, if the transition from state 0 to state 1 occurs at time τ_1, then $\alpha_{\tau_1,1} < \alpha^{\text{lull}}_{\tau_1,1}$. From (15) and (IA.15), the law of motion $\dot{\alpha}_1(\alpha)$ is the same with and without a speculative lull, because $Q_1(\alpha)$ and $\alpha_{1\to2}(\alpha)$ are the same with and without a lull. These results imply $\alpha_{t,1} < \alpha^{\text{lull}}_{t,1}$, for $t \geq \tau_1$. $Q_1(\alpha)$ is assumed in the proposition to be weakly increasing in α. Therefore, $\alpha_{t,1} < \alpha^{\text{lull}}_{t,1}$ implies $\eta_{t,1} \leq \eta^{\text{lull}}_{t,1}$. Next, denote the arrival time of recession by τ_2. By Lemma IA.3 (i.e., $\alpha_{1\to2}(\alpha)$ increasing in α) and using $\alpha_{t,1} < \alpha^{\text{lull}}_{t,1}$, we have that optimists' wealth share at the start of the recession is lower than with a lull $\alpha_{\tau_2,2} < \alpha^{\text{lull}}_{\tau_2,2}$. The law of motion $\dot{\alpha}_2(\alpha) = -\alpha(\lambda^o_{2\to3} - \bar{\lambda}_{2\to3}(\alpha))$ is the same whether or not there was a lull at the start of $s = 1$, thus implying $\alpha_{t,2} < \alpha^{\text{lull}}_{t,2}$ for $t > \tau_2$. Because the recessionary asset price $Q_2(\alpha)$ is increasing in α, the result $\alpha_{t,2} < \alpha^{\text{lull}}_{t,2}$ immediately implies $\eta_{t,2} < \eta^{\text{lull}}_{t,2}$. The welfare cost of the announcement effect ($\Delta w^{pl}_1(\bar{\boldsymbol{\alpha}})$) is the appropriately discounted difference of instantaneous welfare flows with and without a lull at the time when the central bank adopts leaning. Because $W(Q)$ is increasing in Q for $Q < Q^*$, we obtain that, for all $t > \tau_1$ and for every possible history $\mathcal{H}^t$, the instantaneous welfare flow is weakly lower due to the announcement effect, and strictly so during state 2.

We next characterize the magnitude of the welfare losses from the announcement effect. We focus on the size of the welfare losses with small doses of leaning, defined as follows. Many types of leaning policies can be described by the following family of Q-targeting rules: $Q_1(\alpha;\psi)$, with $Q_1(\alpha;\psi)$ strictly decreasing in ψ for all $\alpha \in (0,1)$, and with $Q_1(\alpha;\psi)$ differentiable in α and ψ. That is, higher values of ψ imply more aggressive leaning, with $\psi = 0$ corresponding to conventional policy ($Q_1(\alpha;0) = Q^*$). We say that there are small amounts of leaning with $\psi \approx 0$, in the neighborhood of conventional policy.

Proposition IA.3 (Sowing effects even for small leaning)

Even for small amounts of tightening, the announcement effect implies a first-order welfare loss, as formalized by:

$$\frac{d\Delta w^{pl}_1(\bar{\boldsymbol{\alpha}})}{d\psi}\Big|_{\psi=0} = \frac{dw^{pl}_1(\alpha_{0\to1}(\bar{\boldsymbol{\alpha}}), Q_1(\cdot,\psi))}{d\psi}\Big|_{\psi=0} - \frac{dw^{pl}_1(\bar{\boldsymbol{\alpha}}, Q_1(\cdot,\psi))}{d\psi}\Big|_{\psi=0} < 0.$$

The first-order welfare loss from the announcement effect arises *only* because of lower

recession-state utilization when leaning monetary policy is announced without a speculative lull. Lower utilization during the boom implies only a second-order welfare loss when small doses of leaning monetary policy are used, as formalized by:

$$\frac{dW(Q_1(\alpha_{t,1},\psi))}{d\psi}|_{\psi=0} = \frac{dW(Q_1(\alpha_{t,1}^{\text{lull}},\psi))}{d\psi}|_{\psi=0} = 0, \tag{IA.45}$$

which obtains because the instantaneous welfare function $W(Q)$ is maximized at $Q = Q^*$. Even so, the announcement effect implies a first-order loss in gap value, through utilization during the recession:

$$\underbrace{\frac{dw_2^{pl}(\alpha_{1\to2}(\alpha_{t,1};Q_1(\cdot,\psi)))}{d\psi}|_{\psi=0}}_{\substack{\text{Welfare effect of}\\ \text{leaning without lull}}} < \underbrace{\frac{dw_2^{pl}(\alpha_{1\to2}(\alpha_{t,1}^{\text{lull}};Q_1(\cdot,\psi)))}{d\psi}|_{\psi=0}}_{\substack{\text{Welfare effect of}\\ \text{leaning with lull}}}.$$

With small doses of the leaning, the announcement effect generates first-order welfare losses through lower utilization *during the recession*, whereas the cost of lower utilization *during the boom* is second order. Put differently, under conventional policy, utilization is at its efficient level during the boom (implying small welfare losses from small changes in utilization) but utilization is strictly below its efficient level during the recession.

Proof of Proposition IA.3. Taking the derivative of $\Delta w_1^{pl}(\bar{\boldsymbol{\alpha}})$ with respect to ψ,

$$\frac{dw_1^{pl}(\alpha_{0\to1}(\bar{\boldsymbol{\alpha}}),Q_1(\cdot))}{d\psi} - \frac{dw_1^{pl}(\bar{\boldsymbol{\alpha}};Q_1(\cdot))}{d\psi} = \int_{\tau_1}^{\infty} e^{-(\rho+\lambda_{1\to2}^{pl})(t-\tau_1)}\left[\frac{dW(Q_1(\alpha_{t,1}))}{d\psi} - \frac{dW(Q_1(\alpha_{t,1}^{\text{lull}}))}{d\psi} + \lambda_{1\to2}^{pl}\left(\frac{dw_2^{pl}(\alpha_{1\to2}(\alpha_{t,1};Q_1(\cdot)))}{d\psi} - \frac{dw_2^{pl}(\alpha_{1\to2}(\alpha_{t,1}^{\text{lull}};Q_1(\cdot)))}{d\psi}\right)\right]dt.$$

The welfare function $W(\cdot)$ satisfies $W'(Q^*) = 0$, implying (IA.45). Using the chain rule,

$$\begin{aligned}\frac{dw_2^{pl}(\alpha_{1\to2}(\alpha_{t,1};Q_1(\cdot)))}{d\psi}|_{\psi=0} = &\underbrace{\frac{dw_2^{pl}(\alpha)}{d\alpha}|_{\alpha=\alpha_{1\to2}(\alpha_{t,1};Q^*)}}_{\chi^1}\underbrace{\frac{d\alpha_{1\to2}(\alpha)}{d\alpha}|_{\alpha=\alpha_{t,1},\psi=0}}_{\chi^2}\underbrace{\frac{d\alpha_{t,1}}{d\psi}|_{\psi=0}}_{\chi^3} + \\ &\underbrace{\frac{dw_2^{pl}(\alpha)}{d\alpha}|_{\alpha=\alpha_{1\to2}(\alpha_{t,1};Q^*)}}_{\chi^4}\underbrace{\frac{d\alpha_{1\to2}(\alpha)}{d\psi}|_{\alpha=\alpha_{t,1},\psi=0}}_{\chi^5}.\end{aligned} \tag{IA.46}$$

Similarly,

$$\frac{dw_2^{pl}(\alpha_{1\to 2}(\alpha_{t,1}^{\text{lull}};Q_1(\cdot)))}{d\psi}|_{\psi=0} = \underbrace{\frac{dw_2^{pl}(\alpha)}{d\alpha}|_{\alpha=\alpha_{1\to 2}(\alpha_{t,1}^{\text{lull}};Q^*)}}_{\chi^{1,\text{lull}}} \underbrace{\frac{d\alpha_{1\to 2}(\alpha)}{d\alpha}|_{\alpha=\alpha_{t,1}^{\text{lull}},\psi=0}}_{\chi^{2,\text{lull}}} \underbrace{\frac{d\alpha_{t,1}^{\text{lull}}}{d\psi}|_{\psi=0}}_{\chi^{3,\text{lull}}} + \underbrace{\frac{dw_2^{pl}(\alpha)}{d\alpha}|_{\alpha=\alpha_{1\to 2}(\alpha_{t,1}^{\text{lull}};Q^*)}}_{\chi^{4,\text{lull}}} \underbrace{\frac{d\alpha_{1\to 2}(\alpha)}{d\psi}|_{\alpha=\alpha_{t,1}^{\text{lull}},\psi=0}}_{\chi^{5,\text{lull}}}. \tag{IA.47}$$

There are five derivatives in (IA.46) (labeled $\chi^1, \chi^2, \ldots, \chi^5$) and five related derivatives in (IA.47) (labeled $\chi^{1,\text{lull}}, \chi^{2,\text{lull}}, \ldots, \chi^{5,\text{lull}}$). The proof proceeds by showing the following claims. Claim (i): $\chi^3 < \chi^{3,\text{lull}}$ (or, $\frac{d\alpha_{t,1}}{d\psi}|_{\psi=0} < \frac{d\alpha_{t,1}^{\text{lull}}}{d\psi}|_{\psi=0}$ for $t \geq \tau_1$). Claim (ii): $\chi^1 > 0$ and $\chi^2 > 0$. Claim (iii): $\chi^j = \chi^{j,\text{lull}}$ for $j \in \{1,2,4,5\}$.

The persistent fall in optimists' wealth share due to the announcement effect (Propositions 1 and IA.2) does not become vanishingly small in the neighborhood of conventional policy, in the sense made precise by claim (i). To verify claim (i), note from (29):

$$\frac{d\alpha_{0\to 1}(\alpha)}{d\psi} = \alpha\left(\omega_0^o(\alpha) - 1\right) \frac{Q^*}{Q_1(\alpha_{0\to 1}(\alpha))^2} \frac{\partial Q_1(\alpha_{0\to 1}(\alpha);\psi)}{\partial \psi} \frac{1}{1 - f_{\alpha_1}(\alpha_{0\to 1}(\alpha),\alpha)}. \tag{IA.48}$$

By assumption (see the explanation of Q-targeting rules indexed by ψ in the main text, just before the Proposition), $\frac{dQ_1(\alpha_1;\psi)}{d\psi} < 0$. By the usual argument, $f_{\alpha_1}(\alpha_{0\to 1}(\alpha),\alpha) < 1$ (see Proposition 1). Hence (IA.48), evaluated at $\psi = 0$, is strictly negative. Thus, $\frac{d\alpha_{\tau_1,1}}{d\psi}|_{\psi=0} < 0$ (that is, a marginal increase in ψ, starting from $\psi = 0$, strictly reduces optimists' wealth share when $s = 1$ arrives). In contrast, with a lull, there is no change in optimists' wealth share when $s = 1$ arrives: $\alpha_{0\to 1}^{\text{lull}}(\alpha) = \alpha$, and hence $\frac{d\alpha_{\tau_1,1}^{\text{lull}}}{d\psi} = 0$. Next, define the sensitivities

$$z_t \equiv \left.\frac{d\alpha_{t,1}}{d\psi}\right|_{\psi=0}, \qquad z_t^{\text{lull}} \equiv \left.\frac{d\alpha_{t,1}^{\text{lull}}}{d\psi}\right|_{\psi=0},$$

and let $\bar{\alpha}_t \equiv \alpha_{t,1}|_{\psi=0} = \alpha_{t,1}^{\text{lull}}|_{\psi=0}$ denote the common base path under conventional policy. For $t \geq \tau_1$, both regimes follow the same ODE $\dot{\alpha}_{t,1} = F(\alpha_{t,1};\psi)$ with $F(\alpha;\psi) \equiv \dot{\alpha}_1(\alpha; Q_1(\cdot,\psi))$. Differentiate with respect to ψ and evaluate at $\psi = 0$, in each regime, to obtain:

$$\dot{z}_t = F_\alpha(\bar{\alpha}_t;0)\, z_t + F_\psi(\bar{\alpha}_t;0), \qquad \dot{z}_t^{\text{lull}} = F_\alpha(\bar{\alpha}_t;0)\, z_t^{\text{lull}} + F_\psi(\bar{\alpha}_t;0).$$

Even in a neighborhood of $\psi = 0$, the initial z_t differ because the announcement (with

$\psi > 0$) changes α on impact whereas a lull does not:

$$z_{\tau_1} = \frac{d\alpha_{\tau_1,1}}{d\psi}\Big|_{\psi=0} < 0, \qquad z^{\text{lull}}_{\tau_1} = \frac{d\alpha^{\text{lull}}_{\tau_1,1}}{d\psi}\Big|_{\psi=0} = 0.$$

Let $\Delta z_t \equiv z_t - z^{\text{lull}}_t$. Subtracting the two linear ODEs gives

$$\frac{d\Delta z_t}{dt} = F_\alpha(\bar{\alpha}_t; 0)\,\Delta z_t, \qquad \Delta z_{\tau_1} < 0,$$

hence

$$\Delta z_t = \Delta z_{\tau_1} \exp\left(\int_{\tau_1}^{t} F_\alpha(\bar{\alpha}_u; 0)\,du\right) < 0 \quad \forall t \geq \tau_1.$$

Therefore $z_t < z^{\text{lull}}_t$ for all $t \geq \tau_1$, i.e., $\chi^3 < \chi^{3,\text{lull}}$, which establishes claim (i).

For claim (ii), we will show $\frac{dw^{pl}_2(\alpha)}{d\alpha} > 0$, for $\alpha \in (0,1)$ and also $\frac{d\alpha_{1\to2}(\alpha)}{d\alpha}|_{\alpha=\alpha_{t,1},\psi=0} > 0$. We can write the gap value in $s = 2$ as

$$w^{pl}_2(\alpha) = \int_0^\infty e^{-(\rho+\lambda^{pl}_{2\to3})t}\left(W(Q_2(\alpha_{t,2})) + \lambda^{pl}_{2\to3} w^{pl}_3(\alpha_{2\to3}(\alpha_{t,2}))\right) dt, \tag{IA.49}$$

where $\alpha_{t,2}$ satisfies $\dot{\alpha}_{t,2} = \alpha_{t,2}(1-\alpha_{t,2})(\lambda^{p}_{2\to3} - \lambda^{o}_{2\to3})$ and the initial value $\alpha_{t=0,2} = \alpha$. Noting that $w^{pl}_3(\alpha) = 0$ for $\alpha \in [0,1]$, and taking the derivative of (IA.49) with respect to α,

$$\frac{dw^{pl}_2(\alpha)}{d\alpha} = \int_0^\infty e^{-(\rho+\lambda^{pl}_{2\to3})t} \frac{dW(Q_2(\alpha_{t,2}))}{d\alpha} dt > 0,$$

where the inequality obtains because $\frac{dW(Q_2(\alpha_{t,2}))}{d\alpha} > 0$ for all $\alpha_{t,2} \in [0,1]$. The second part of Claim (ii), $\frac{d\alpha_{1\to2}(\alpha)}{d\alpha}|_{\alpha=\alpha_{t,1},\psi=0} > 0$, follows from Lemma IA.3.

For claim (iii), observe that the mappings from optimists' wealth share to the $s = 2$ gap value and the change in optimists' wealth share when $s = 2$ arrives ($w^{pl}_2(\alpha)$ and $\alpha_{1\to2}(\alpha)$) are the same with and without a lull. Hence, evaluated at $\psi = 0$ (or equivalently $Q_1(\alpha) = Q^*$), $\chi^j = \chi^{j,\text{lull}}$ for $j \in \{1,2,4,5\}$.

D.2 Timing, the announcement effect, and optimal policy

This subsection quantifies how the effects of leaning depend on *when* the policy is introduced and on the sequence of state transitions that occur beforehand. Timing matters quantitatively in our model because delaying the introduction of leaning changes optimists' wealth share, α (the key state variable). Because α drifts within each state and jumps at state transitions, shifting either the start of the boom or the date at which leaning is introduced changes the optimists' wealth share at the time of announcement, $\bar{\alpha}$. Such shifts therefore change (i) the size of the announcement effect under simple

leaning rules and (ii) the intensity and welfare consequences of (approximately) optimal leaning.

A quantitative timing analysis. To conduct the timing analysis, we modify the benchmark environment of Section 1 so that the economy begins in a bust and then enters a boom before leaning is introduced.[14] Concretely, we add an initial bust state $s = -1$ prior to the usual $s = 0$ boom. This initial bust has the same low growth rate as the benchmark bust, $g_{-1} = g_2$, and the same transition arrival rates, $\lambda^i_{-1\to 0} = \lambda^i_{2\to 3}$ for all i. Monetary policy in $s = -1$ is Wicksellian but constrained by the ZLB (under the same parametric restriction on g_2 that implies a negative neutral rate in the $s = 2$ bust).

Let τ_0 denote the time at which the economy transitions from the initial bust into the boom ($s = 0$), and let $\tau_1 > \tau_0$ denote the date at which leaning is introduced (the arrival of state $s = 1$). Define the pre-announcement boom duration as $T \equiv \tau_1 - \tau_0$.

Table IA.II reports outcomes as we vary either (i) the initial bust length τ_0, holding fixed T, or (ii) the pre-announcement boom duration T, holding fixed τ_0. When varying τ_0, we hold the time spent in the $s = 0$ boom fixed at $1/\lambda^{pl}_{0\to 2}$, the expected boom duration under the planner's belief. When varying T, we instead hold the initial bust length fixed at $1/\lambda^{pl}_{-1\to 0}$, the expected bust duration under the planner's belief. Throughout, the initial optimists' wealth share at $t = 0$ is 0.8.

For each timing scenario, we report three objects that correspond closely to the discussion in the main text. First, for a simple leaning rule, we report the announcement effect, measured by the on-impact change in the asset price at τ_1. Second, we report the asset-price implications under the (approximation to the) optimal rule, which, by design, avoids an on-impact drop and instead reaches the "optimal-lull" price only after the phase-in window. Third, we report the associated welfare gain relative to the Wicksellian benchmark.

Policies and reported outcomes. We consider two Q-targeting rules in state $s = 1$. The first, $Q_1^{\text{simple}}(\cdot)$, is the simple (non-optimal) rule in Eq. (28) with $\psi = 0.5$, which is also used in main-text Figures 1–3. The second, $Q_1^{\text{optimal}}(\cdot)$, is the approximation to the optimal rule in Eq. (35), for the small value of $\hat{\alpha}$ used in the main-text analyses.

Under the simple rule, the announcement effect is reflected in the contemporane-

[14] We thank the editor for this suggestion.

Table IA.II: **Timing of leaning: Implications under simple and optimal policies**

Panel A. Varying time in $s=-1$ bust				
	Announcement effect		Optimal leaning	Welfare gain
Time in $s=-1$ bust	Simple	Optimal	After phase-in	vs. Wicksellian
τ_0	$\frac{Q_1^{\text{simple}}(\alpha_{\tau_1,1})}{Q^*}-1$	$\frac{Q_1^{\text{optimal}}(\alpha_{\tau_1,1})}{Q^*}-1$	$\frac{Q_1^{\text{optimal}}(\alpha_{\tau_1,1}+\hat{\alpha})}{Q^*}-1$	$\Delta w^{pl,\text{optimal}}(\alpha_{\tau_1,1})$
0.8	-0.008	0	-0.005	0.083
1.0	-0.010	0	-0.012	0.124
1.5	-0.015	0	-0.017	0.199
3.0	-0.057	0	-0.005	0.079
Panel B. Varying time in $s=0$ boom				
	Announcement effect		Optimal leaning	Welfare gain
Time in $s=0$ boom	Simple	Optimal	After phase-in	vs. Wicksellian
$\tau_1-\tau_0$	$\frac{Q_1^{\text{simple}}(\alpha_{\tau_1,1})}{Q^*}-1$	$\frac{Q_1^{\text{optimal}}(\alpha_{\tau_1,1})}{Q^*}-1$	$\frac{Q_1^{\text{optimal}}(\alpha_{\tau_1,1}+\hat{\alpha})}{Q^*}-1$	$\Delta w^{pl,\text{optimal}}(\alpha_{\tau_1,1})$
2.5	-0.013	0	-0.015	0.180
5.0	-0.010	0	-0.012	0.124
6.0	-0.009	0	-0.009	0.101
7.5	-0.008	0	-0.002	0.062

ous drop in the asset price at τ_1, $\frac{Q_1^{\text{simple}}(\alpha_{\tau_1,1})}{Q^*}-1$, reported in the second column. This on-impact drop is accompanied by an endogenous decline in optimists' wealth share upon announcement, i.e., $\alpha_{\tau_1,1}<\bar{\boldsymbol{\alpha}}$. In contrast, under optimal leaning there is no contemporaneous asset-price drop at τ_1, $\frac{Q_1^{\text{optimal}}(\alpha_{\tau_1,1})}{Q^*}-1=0$, as reported in the third column. Instead, the asset price is reduced gradually during the phase-in window and reaches the target "optimal-lull" level only after phase-in, $\frac{Q_1^{\text{optimal}}(\alpha_{\tau_1,1}+\hat{\alpha})}{Q^*}-1$, reported in the fourth column. The fifth column reports the welfare gain under optimal policy relative to Wicksellian policy, $\Delta w^{pl,\text{optimal}}(\alpha_{\tau_1,1})$.

Summary of results. The table illustrates three simple quantitative patterns.

- **Simple leaning rule.** A longer initial bust (higher τ_0) strengthens the announcement effect, while a longer pre-announcement boom (higher T) attenuates it.
- **Optimal leaning.** Within the scenarios studied, delaying the introduction of optimal leaning within the boom (higher T) implies *less aggressive* optimal leaning after phase-in (a smaller decline in Q).

- **Welfare.** Delaying the introduction of optimal leaning within the boom (higher T) reduces the welfare gains relative to Wicksellian policy.

All three patterns follow from the same mechanism: by changing $\bar{\alpha}$ at the time leaning is introduced, shifts in timing move the economy to a different region of the state space, which changes both the announcement dynamics under simple leaning and the strength of the macroprudential motive under optimal leaning.

Panel A: varying the initial bust length. Panel A varies the time spent in the initial bust state $s = -1$, τ_0, holding fixed the time spent in the $s = 0$ boom before leaning is introduced, $T = \tau_1 - \tau_0$. A longer bust reduces optimists' wealth share before the policy change, so the economy enters the leaning regime with a lower $\bar{\alpha}$. Under the *simple* leaning rule, this makes the announcement effect larger: the on-impact drop in Q at τ_1 increases in magnitude from -0.008 to -0.057 as τ_0 rises from 0.8 to 3.0 (second column).

Under *optimal* leaning, there is no on-impact drop at announcement (third column). The relevant object is therefore the decline in Q after phase-in (fourth column), which is non-monotone in τ_0: optimal leaning is strongest for intermediate initial-bust lengths and weaker for very short or very long busts. The welfare gains in the fifth column mirror this pattern. They are inverted-U-shaped in the initial bust length, peaking at intermediate values of τ_0 (and hence intermediate values of $\bar{\alpha}$). This is consistent with the main-text result that the macroprudential benefits from leaning are strongest at intermediate optimist wealth shares: when the economy is close to an all-optimist or all-pessimist limit, α changes little in the downturn and the marginal benefit of leaning is smaller.

Panel B: varying time spent in the boom before the policy change. Panel B varies the pre-announcement boom duration $T = \tau_1 - \tau_0$, holding fixed the initial bust length τ_0. A longer $s = 0$ boom increases optimists' wealth share prior to announcement, so the leaning regime is introduced at a higher $\bar{\alpha}$.

This higher $\bar{\alpha}$ attenuates the announcement effect under the simple leaning rule: the on-impact decline in Q becomes smaller in magnitude as T rises (second column). Under optimal leaning, the absence of an on-impact price drop again shifts attention to the post-phase-in target (fourth column). Within the scenarios shown, a longer pre-announcement boom implies less aggressive optimal leaning after phase-in: the optimal

asset price is closer to Q^* (i.e., the decline is smaller in magnitude). Finally, Panel B shows that welfare gains from optimal leaning decline as T increases (fifth column). Delaying optimal leaning within the boom moves the economy toward a region with a weaker incentive to lean (because $\bar{\alpha}$ is higher within the range spanned by these scenarios).[15]

E. Proofs and derivations: Extensions

E.1 Endogenous bust probability

Let $\mathbb{Q}$ denote the set of admissible Q-targeting rules that are continuous and right-differentiable and satisfy the cure-no-worse-than-the-disease constraint. For $i \in \{o, p, pl\}$, the bust multiplier $\gamma^i(\cdot)$ is a mapping from $\mathbb{Q}$ into $\mathbb{R}^+$. We assume that, for $Q_1(\cdot) \in \mathbb{Q}$, optimists continue to perceive a lower recession arrival rate ($\gamma^o(Q_1(\cdot))\lambda^o_{1\to 2} < \gamma^p(Q_1(\cdot))\lambda^p_{1\to 2}$).

For a given rule $Q_1(\cdot)$, we denote the collection of multipliers by $\Gamma \equiv (\gamma^o, \gamma^p, \gamma^{pl}) \in \mathbb{R}^3_+$. We denote the announcement effect by $\alpha_{0\to 1}(\alpha; \Gamma)$, optimists' wealth share upon recession by $\alpha_{1\to 2}(\alpha; \Gamma)$, and the growth rate of optimists' wealth share by $\dot{\alpha}_1(\alpha; \Gamma)$. For the benchmark, we retain the notation used thus far (e.g., $\alpha_{1\to 2}(\alpha)$ for optimists' share upon recession).

The next result summarizes how an endogenous bust probability affects the benefits and costs of a given leaning policy.

Proposition IA.4 (Endogenous bust probability and leaning) *Fix a rule* $Q_1(\cdot)$.

Announcement effect. *Relative to the benchmark of exogenous bust risk, the fall in optimists' wealth share upon announcement of* $Q_1(\cdot)$ *is unchanged:* $\alpha_{0\to 1}(\alpha; \Gamma) = \alpha_{0\to 1}(\alpha)$.

Macroprudential benefit and cost of leaning in $s = 1$. *Suppose that either optimists and pessimists agree about the endogenous bust multiplier* ($\gamma^o(Q_1(\cdot)) = \gamma^p(Q_1(\cdot))$) *or the leverage constraint is binding for optimists under both exogenous and endogenous bust risk. Then, in the presence of endogenous bust risk, the fall in optimists' wealth share upon*

[15] In Panel B the welfare gain from optimal policy is monotonically decreasing in T, whereas Panel A displays a non-monotone pattern in τ_0. The difference reflects the implied values of $\bar{\alpha}$ at announcement across the scenarios: in Panel B, $\bar{\alpha}$ remains high in all four cases because the initial bust length is fixed at its expected value under the planner's belief, while in Panel A varying the bust length moves $\bar{\alpha}$ across a broader range. The dependence of optimal leaning and welfare gains on $\bar{\alpha}$ is summarized in main-text Figure 5.

recession is unchanged relative to the benchmark of exogenous bust risk,

$$\alpha_{1\to 2}(\alpha;\Gamma) = \alpha_{1\to 2}(\alpha), \tag{IA.50}$$

and the growth rate of optimists' wealth share is lower if and only if the targeting rule reduces the arrival rate of the bust under pessimists' beliefs:

$$\dot{\alpha}_1(\alpha;\Gamma) < \dot{\alpha}_1(\alpha) \text{ if and only if } \gamma^p(Q_1(\cdot)) < 1. \tag{IA.51}$$

A cost of some (non-optimal) leaning policies is a fall in optimists' wealth share upon announcement. The magnitude of this fall is unchanged with endogenous bust risk because (for a given $Q_1(\cdot)$ rule) the fall in optimists' wealth share (Eq. (29)) is determined solely by $s = 0$ leverage, which is unaffected by bust arrival rates in $s = 1$. Leverage in $s = 0$ is unaffected because the announcement of leaning in $s = 1$ is unexpected. Below, we also show that if some leaning is allowed in $s = 0$, then the presence of endogenous bust risk in $s = 0$ still leaves unchanged the magnitude of the fall in optimists' wealth share upon announcement if optimists and pessimists agree about the bust multipliers ($\gamma^o = \gamma^p$).

A benefit of leaning is to alleviate the fall in optimists' wealth share upon recession. This fall in optimists' wealth share (conditional on a $Q_1(\cdot)$ targeting rule) is determined solely by $s = 1$ leverage (Eq. (13)). Optimists' desired leverage is a function of the *ratio* of optimists' belief about the bust arrival rate to pessimists' belief. Therefore, with homogeneous multipliers ($\gamma^o = \gamma^p$) or a binding leverage limit, leverage in $s = 1$ and the fall in optimists' wealth share upon recession are unaffected by endogenous bust risk, as formalized in Eq. (IA.50). That is, the macroprudential benefit of leaning is unchanged.

Proposition IA.4 sheds light on the welfare implications of leaning with an endogenous bust probability. If optimists' wealth share is $\bar{\alpha}$ prior to leaning's adoption, then the gap value associated with targeting rule $Q_1(\cdot)$ is $w_1^{pl}(\alpha_{0\to 1}(\bar{\alpha}),\Gamma)$ with

$$\rho w_1^{pl}(\alpha;\Gamma) = W(Q_1(\alpha)) + \underbrace{\gamma^{pl}(Q_1(\cdot))\lambda_{1\to 2}^{pl}}_{\text{Arrival rate of bust}} \underbrace{\left(w_2^{pl}(\alpha_{1\to 2}(\alpha;\Gamma)) - w_1^{pl}(\alpha;\Gamma)\right)}_{\text{Gap value loss upon bust}} + \underbrace{\frac{dw_1^{pl}(\alpha,\Gamma)}{d\alpha}\dot{\alpha}_1(\alpha;\Gamma)}_{\text{Wealth share growth}}.$$

Any endogenous change in the planner's bust arrival rate $\gamma^{pl}(Q_1(\cdot))\lambda_{1\to 2}^{pl}$ directly affects the gap value; a lower bust arrival rate increases the boom's expected length. With an endogenous bust probability, the announcement effect may still be present ($\alpha_{0\to 1}(\alpha;\Gamma) <$

α if $Q_1(\bar{\boldsymbol{\alpha}}) < Q^*$). Per Proposition IA.4, the fall in optimists' wealth share upon recession $\alpha_{1\to 2}(\alpha;\Gamma)$ is unchanged under homogeneous multipliers or if the leverage limit binds. In addition, the endogenous bust probability may affect the growth rate of optimists' share.

To prove this proposition, we define some additional notation. For $i \in \{o, p, pl\}$, define the endogenous bust rate

$$\tilde{\lambda}^i_{1\to 2} \;\equiv\; \gamma^i(Q_1(\cdot))\,\lambda^i_{1\to 2}. \tag{IA.52}$$

Investors' problem and the equilibrium characterization in Section 1 remain valid after the substitution of $\lambda^i_{1\to 2}$ with $\tilde{\lambda}^i_{1\to 2}$.

Proof of Proposition IA.4. When the leverage limit is slack in $s = 1$, the post-bust optimist share (from (16)) is:

$$\alpha^{\text{unc}}_{1\to 2}(\alpha;\Gamma) = \alpha\frac{\tilde{\lambda}^o_{1\to 2}}{\tilde{\lambda}^o_{1\to 2}\,\alpha + \tilde{\lambda}^p_{1\to 2}\,(1-\alpha)}. \tag{IA.53}$$

Optimists' desired leverage, $\omega_1^{o,unc}(\alpha;\Gamma)$, solves (from (13)),

$$\frac{\alpha^{\text{unc}}_{1\to 2}(\alpha;\Gamma)}{\alpha} = 1 - \left(\omega_1^{o,\text{unc}}(\alpha;\Gamma) - 1\right)\left(\frac{Q_1(\alpha)}{Q_2(\alpha^{\text{unc}}_{1\to 2}(\alpha;\Gamma))} - 1\right). \tag{IA.54}$$

Taking account of the leverage limit, optimists' wealth share upon state transition satisfies

$$\frac{\alpha_{0\to 1}(\alpha;\Gamma)}{\alpha} = 1 - \left(\min\{\bar{\omega}, \omega_0^{o,\text{unc}}(\alpha;\Gamma)\} - 1\right)\left(\frac{Q^*}{Q_1(\alpha_{0\to 1}(\alpha;\Gamma))} - 1\right), \tag{IA.55}$$

$$\frac{\alpha_{1\to 2}(\alpha;\Gamma)}{\alpha} = 1 - \left(\min\{\bar{\omega}, \omega_1^{o,\text{unc}}(\alpha;\Gamma)\} - 1\right)\left(\frac{Q_1(\alpha)}{Q_2(\alpha_{1\to 2}(\alpha;\Gamma))} - 1\right). \tag{IA.56}$$

Announcement effect. Conditional on a given $Q_1(\cdot)$ rule, Eq. (IA.55) depends on $\omega_0^{o,\text{unc}}(\alpha;\Gamma)$, which is determined by beliefs about transition from $s = 0$ to $s = 2$ ($\lambda^i_{0\to 2}$). Because $s = 1$ is unexpected, the perceived bust rates in $s = 1$ do not enter agents' $s = 0$ portfolio problem. Therefore, $\omega_0^{o,\text{unc}}(\alpha;\Gamma) = \omega_0^{o,\text{unc}}(\alpha)$, and the fixed point (IA.55) coincides with the benchmark:

$$\alpha_{0\to 1}(\alpha;\Gamma) = \alpha_{0\to 1}(\alpha). \tag{IA.57}$$

This proves the "Announcement effect" part of the proposition. (Remark: In our benchmark model, we assume policy is Wicksellian in $s = 0$. If we instead allow some leaning in $s = 0$ ($Q_0(\alpha) < Q^*$), with endogenous bust rates $\tilde{\lambda}^i_{0\to 2}$, desired leverage in $s = 0$ is unchanged under homogeneous multipliers. To see this, replace $s = 1$ with $s = 0$ in (IA.53). Hence, (IA.57) still holds for the unexpected announcement of $Q_1(\alpha)$ at the start

of $s = 1$.)

Invariance of bust drop under homogeneous multipliers or binding leverage limit. Fix $\alpha \in (0,1)$. If $\gamma^o = \gamma^p \equiv \gamma$, then desired leverage in $s = 1$ is unaffected by the endogenous bust multiplier, because (IA.53) depends on the *ratio* $\tilde{\lambda}^o_{1\to 2}/\tilde{\lambda}^p_{1\to 2} = \lambda^o_{1\to 2}/\lambda^p_{1\to 2}$ ((IA.52)–(IA.54)). Combining this observation with (IA.56), we obtain

$$\alpha_{1\to 2}(\alpha;\Gamma) = \alpha_{1\to 2}(\alpha). \tag{IA.58}$$

Alternatively, if the leverage limit binds at $(1,\alpha)$ both with exogenous and endogenous bust arrival rates, then $\min\{\bar{\omega}, \omega_1^{o,\text{unc}}(\alpha;\cdot)\} = \bar{\omega}$ in both cases, and (IA.56) again yields (IA.58). This establishes (IA.50).

Boom-time growth. For $s = 1$, wealth-share growth satisfies

$$\frac{\dot{\alpha}_1(\alpha;\Gamma)}{\alpha} = \tilde{\lambda}^p_{1\to 2}\,\frac{1-\alpha}{1-\alpha_{1\to 2}(\alpha;\Gamma)}\left(1 - \frac{\alpha_{1\to 2}(\alpha;\Gamma)}{\alpha}\right).$$

Using (IA.58) and the definition $\tilde{\lambda}^p_{1\to 2} = \gamma^p \lambda^p_{1\to 2}$, we obtain that (IA.51) holds if $\gamma^o = \gamma^p$, or if the leverage limit binds under both exogenous and endogenous bust risk.

Heterogeneous multipliers with slack leverage limit. In the case of heterogeneous bust multipliers ($\gamma^o \neq \gamma^p$) and slack leverage limits, the fall in optimists' wealth share upon recession has a larger magnitude if and only if belief dispersion increases: $\alpha_{1\to 2}(\alpha;\Gamma) < \alpha_{1\to 2}(\alpha)$ if and only if $\gamma^o < \gamma^p$. Greater belief dispersion drives a higher desired leverage for optimists. Hence, if leverage constraints are slack, the fall in optimists' wealth share upon recession is amplified if endogenous bust multipliers make optimists even more optimistic relative to pessimists.

To see this, suppose $\gamma^o \neq \gamma^p$ and the cap is slack in $s = 1$ (hence, $\alpha_{1\to 2}(\alpha;\Gamma) = \alpha^{unc}_{1\to 2}(\alpha;\Gamma)$). Fix $\alpha \in (0,1)$. From Eq. (IA.52), $\alpha_{1\to 2}(\alpha;\Gamma)$ is increasing in γ^o and decreasing in γ^p. Using Eq. (IA.53),

$$\alpha^{unc}_{1\to 2}(\alpha;\Gamma) = \frac{\alpha}{\alpha + \frac{\lambda^p_{1\to 2}}{\lambda^o_{1\to 2}}\frac{\gamma^p}{\gamma^o}(1-\alpha)}.$$

Because the benchmark with exogenous bust risk corresponds to $\gamma^o = \gamma^p = 1$, it follows that $\alpha_{1\to 2}(\alpha;\Gamma) < \alpha_{1\to 2}(\alpha)$ if and only if $\gamma^o < \gamma^p$.

Optimal policy with endogenous bust probability. We next describe the planner's problem with endogenous bust multipliers, under the assumptions about multipliers

described in Section 5.1. The planner chooses $Q_1(\cdot)$ to maximize the gap value rate, or the gap value divided by the expected length of time until recovery. The gap value rate corresponds to the average (discounted) expected welfare loss from under-utilization.[16] The expected time until recovery is $(\tilde{\lambda}^{pl}_{1\to 2}(Q_1(\cdot)))^{-1} + (\lambda^{pl}_{2\to 3})^{-1}$.

The set of Q-targeting rules available to the planner can be divided into three classes.

First, the rule can be Wicksellian (i.e., $Q_1(\alpha) = Q^*, \forall \alpha$), which would result in no change in bust arrival rates (recall that $\gamma^o = \gamma^p = \gamma^{pl} = 1$ for Wicksellian). Let $w_1^{pl,Wick}(\alpha)$ denote the $s = 1$ gap value rate conditional on maintaining Wicksellian policy, with exogenous bust arrival rates $\lambda^i_{1\to 2}$.

Second, the rule can satisfy $Q_1(\bar{\boldsymbol{\alpha}}) = Q^*$ and lean $Q_1(\alpha) < Q^*$ for some $\alpha > \bar{\boldsymbol{\alpha}}$ (while also being continuous and right-differentiable and satisfying cure-no-worse-than-the-disease). If so, the bust arrival rates are $\tilde{\lambda}^i_{1\to 2} < \lambda^i_{1\to 2}$. The constraint $Q_1(\bar{\boldsymbol{\alpha}}) = Q^*$ is satisfied by (35). Hence, within this second class of policies, the optimal policy is characterized by Proposition 2, substituting $\tilde{\lambda}^i_{1\to 2}$ for $\lambda^i_{1\to 2}$, so long as $Q^{\text{lull}}(\alpha) < Q^*$ for some positive-measure set of α. Let $\hat{w}_1^{pl}(\alpha)$ denote the gap value rate conditional on this Q-targeting rule. These leaning policies involve gradually reducing the asset price, per Proposition 2.

Third, the rule can satisfy $Q_1(\bar{\boldsymbol{\alpha}}) < Q^*$ (while also being continuous and right-differentiable and satisfying cure-no-worse-than-the-disease). If so, the bust arrival rates are $\tilde{\lambda}^i_{1\to 2} > \lambda^i_{1\to 2}$. Define $\hat{\hat{w}}_1^{pl}(\alpha)$ as the gap value rate conditional on the best Q-targeting rule within this class. The solution to the planner's problem with endogenous bust multipliers is given by $\max\{w_1^{pl,Wick}(\alpha), \hat{w}_1^{pl}(\alpha), \hat{\hat{w}}_1^{pl}(\alpha)\}$. In our quantitative analysis, as expected, the maximum is $\hat{w}_1^{pl}(\alpha)$: Optimal policy leans by gradually reducing the asset price.

E.2 Good booms, bad booms

We extend the model to allow for a state having two successor states, with investors ascribing a strictly positive transition rate for each of the two successor states. We then use these characterizations to study "good booms" and "bad booms."

[16] With exogenous bust arrival rates, the ratio of the gap value rate to the gap value is invariant to policy. This invariance no longer applies when the bust arrival rate (under the planner's belief) is endogenous and depends on policy.

E.2.1 Investors' problem and equilibrium with two successor states

We say that s' is a successor state of s if $\lambda^i_{s\to s'} > 0$. In the benchmark model of the main text, for each state, there is at most one successor state. We now extend the model to allow for two successor states.

Description of successor states. Suppose that from any state s, there is the possibility of transition to state s' with arrival rate $\lambda^i_{s\to s'}$ and to s'' with arrival rate $\lambda^i_{s\to s''}$. If $\lambda^i_{s\to s'} = \lambda^i_{s\to s''} = 0$, then s is an absorbing state. If $\lambda^i_{s\to s'} > 0$ but $\lambda^i_{s\to s''} = 0$, then s has one successor state. If $\lambda^i_{s\to s'} > 0$ and $\lambda^i_{s\to s''} > 0$, then s has two successor states.

Financial markets and investors' problem. We assume there is an Arrow-Debreu security in zero net supply that pays out 1 unit of the consumption good upon transition from s to s''. This security has price $p^{s''}_{t,s}$. Investors continue to trade claims on all output, with $Q_{t,s}$ the price per unit of capital. With two successor states, this set of assets is sufficient for markets to be dynamically complete when the leverage constraint (on the output claims) does not bind. Continue to denote by $\omega^i_{t,s}$ the value of investor i's claims on output as a share of investor i's wealth. Denote by $\omega^{i,s''}_{t,s}$ the flow expenditure on the Arrow-Debreu security that pays out 1 unit of the consumption good in the event of state transition to state s''.

The problem of an investor of type i at time t in state s is:

$$V^i_{t,s}(a^i_{t,s}) = \max_{\{c^i_{\tau,\tilde{s}},\omega^i_{\tau,\tilde{s}},\omega^{i,s''}_{\tau,\tilde{s}}\}_{\tau\geq t,\tilde{s}}} \mathbb{E}^i_{t,s}\left[\int_t^\infty e^{-\rho(\tau-t)}\log c^i_{\tau,\tilde{s}}d\tau\right]$$

subject to, for all possible s and t, absent transition,

$$da^i_{t,s} = \left(a^i_{t,s}(r^f_{t,s} + \omega^i_{t,s}(r_{t,s} - r^f_{t,s}) - \omega^{i,s''}_{t,s}) - c^i_{t,s}\right)dt,$$

and, upon transition from s to s',

$$a^i_{t,s'} = a^i_{t,s}\left(1 + \omega^i_{t,s}\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}\right), \tag{IA.59}$$

and, upon transition from s to s'',

$$a^i_{t,s''} = a^i_{t,s}\left(1 + \omega^i_{t,s}\frac{Q_{t,s''} - Q_{t,s}}{Q_{t,s}} + \omega^{i,s''}_{t,s}\frac{1}{p^{s''}_{t,s}}\right), \tag{IA.60}$$

as well as the leverage constraint $\omega^i_{t,s} \leq \bar{\omega}_s$. Note that $a^i_{t,s}\omega^{i,s''}_{t,s}/p^{s''}_{t,s}$ is the number of Arrow-Debreu claims purchased by the investor.

The HJB equation is:

$$\rho V^i_{t,s}(a^i_{t,s}) = \max_{c,\omega,\omega^{s''}} \log(c) + \frac{\partial V^i_{t,s}}{\partial a}\left(a^i_{t,s}\left(r^f_{t,s} + \omega(r_{t,s} - r^f_{t,s}) - \omega^{s''}\right) - c\right) + \\ + \frac{\partial V^i_{t,s}(a^i_{t,s})}{\partial t} + \lambda^i_{s\to s'}\left(V^i_{t,s'}\left(a^i_{t,s}\left(1 + \omega\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}\right)\right) - V^i_{t,s}(a^i_{t,s})\right) + \\ + \lambda^i_{s\to s''}\left(V^i_{t,s''}\left(a^i_{t,s}\left(1 + \omega\frac{Q_{t,s''} - Q_{t,s}}{Q_{t,s}} + \frac{\omega^{s''}}{p^{s''}_{t,s}}\right)\right) - V^i_{t,s}(a^i_{t,s})\right).$$

The first-order condition for ω is:

$$\frac{\partial V^i_{t,s}}{\partial a}(r_{t,s} - r^f_{t,s}) + \lambda^i_{s\to s'}\frac{\partial V^i_{t,s'}(a^i_{t,s'})}{\partial a}\left(\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}\right) + \lambda^i_{s\to s''}\frac{\partial V^i_{t,s''}(a^i_{t,s''})}{\partial a}\left(\frac{Q_{t,s''} - Q_{t,s}}{Q_{t,s}}\right) \geq 0. \tag{IA.61}$$

The first-order condition for $\omega^{s''}$ is:

$$-\frac{\partial V^i_{t,s}}{\partial a} + \lambda^i_{s\to s''}\frac{\partial V^i_{t,s''}(a^i_{t,s''})}{\partial a}\frac{1}{p^{s''}_{t,s}} = 0. \tag{IA.62}$$

As in the economy with only one successor state, $V^i_{t,s}(a^i_{t,s}) = \frac{1}{\rho}\log(\frac{a^i_{t,s}}{Q_{t,s}}) + v^i_{t,s}$, implying

$$\frac{\partial V^i_{t,s}(a^i_{t,s})}{\partial a} = \frac{1}{\rho a^i_{t,s}}. \tag{IA.63}$$

Combining (IA.62) and (IA.63), we have

$$\frac{p^{s''}_{t,s}}{\lambda^i_{s\to s''}} = \frac{\frac{\partial V^i_{t,s''}(a^i_{t,s''})}{\partial a}}{\frac{\partial V^i_{t,s}}{\partial a}} = \frac{a^i_{t,s}}{a^i_{t,s''}}, \tag{IA.64}$$

where the last equality above uses the result that, as with only one successor state, Eqs. (IA.2)–(IA.3) hold here as well.

Beliefs about transition rate to s''. We assume that investors agree about the transition rate from s to s'', with $\lambda^o_{s\to s''} = \lambda^p_{s\to s''}$. This is consistent with the analysis of "good booms" in Section 5.2 and Appendix E.2.2. One implication is $\alpha^i_{t,s} = \alpha^i_{t,s''}$ (wealth shares do not change upon transition to s'').

Portfolio optimality condition for the two-successor state problem. The portfolio-optimality condition with one successor state is Eq. (14). To derive an analogous condition for the two-successor-state problem, we combine the first-order condition for ω, Eq. (IA.61), with Eq. (IA.64) and $a^i_{t,s} = \alpha^i_{t,s}Q_{t,s}k_{t,s}$ to obtain:

$$r_{t,s} + \lambda^i_{s\to s''}\left(\frac{Q_{t,s''} - Q_{t,s}}{Q_{t,s''}}\right) + \lambda^i_{s\to s'}\frac{\alpha^i_{t,s}}{\alpha^i_{t,s'}}\left(\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}}\right) \geq r^f_{t,s}, \tag{IA.65}$$

which holds with equality if the leverage limit is not binding for investor i. This portfolio-

optimality condition is identical to the one-state version, Eq. (14), except that the expected risk-adjusted return on the left side includes a term for the risk-adjusted capital gain upon transition to s'', or $\lambda^i_{s\to s''}\left(\frac{Q_{t,s''}-Q_{t,s}}{Q_{t,s''}}\right)$. Because investors agree about the transition rate from s to s'', the risk-adjustment, $\frac{\frac{\partial V^i_{t,s''}(a^i_{t,s''})}{\partial a}}{\frac{\partial V^i_{t,s}}{\partial a}} = \frac{a^i_{t,s}}{a^i_{t,s''}}$, simplifies to $\frac{Q_{t,s}}{Q_{t,s''}}$.

Trading in the Arrow-Debreu security. Because wealth shares do not change upon transition to state s'', Eq. (IA.60) implies

$$\left(\omega^o_{t,s}-\omega^p_{t,s}\right)\frac{Q_{t,s''}-Q_{t,s}}{Q_{t,s}} = (\omega^{p,s''}_{t,s}-\omega^{o,s''}_{t,s})\frac{1}{p^{s''}_{t,s}}. \tag{IA.66}$$

For the same reason, upon transition to s'', the wealth of each investor type changes by the same factor as aggregate wealth:

$$\frac{a^i_{t,s''}}{a^i_{t,s}} = \frac{Q_{t,s''}}{Q_{t,s}}. \tag{IA.67}$$

Combining (IA.64) and (IA.67), we obtain

$$p^{s''}_{t,s} = \lambda^i_{s\to s''}\frac{Q_{t,s}}{Q_{t,s''}}.$$

Substituting into (IA.66) and rearranging, one obtains:

$$\lambda^i_{s\to s''}\left(\omega^o_{t,s}-\omega^p_{t,s}\right)\left(1-\frac{Q_{t,s}}{Q_{t,s''}}\right) = \omega^{p,s''}_{t,s}-\omega^{o,s''}_{t,s}. \tag{IA.68}$$

Market clearing for the derivative requires: $\alpha^o_{t,s}\omega^{o,s''}_{t,s}+\alpha^p_{t,s}\omega^{p,s''}_{t,s}=0$. Equivalently, $\omega^{p,s''}_{t,s}-\omega^{o,s''}_{t,s} = \frac{1}{\alpha^o_{t,s}}\omega^{p,s''}_{t,s}$. Market clearing for the claim on output requires

$$\omega^o_{t,s}-\omega^p_{t,s} = \frac{1}{\alpha^o_{t,s}}(1-\omega^p_{t,s}).$$

Substituting into (IA.68),

$$\lambda^p_{s\to s''}(1-\omega^p_{t,s})\left(1-\frac{Q_{t,s}}{Q_{t,s''}}\right) = \omega^{p,s''}_{t,s}. \tag{IA.69}$$

Optimist wealth share growth, $\dot\alpha_{t,s}$. Market clearing for financial assets and $c^i_{t,s}=\rho a^i_{t,s}$ imply $r_{t,s}-\rho = \frac{d(Q_{t,s}k_{t,s})/dt}{Q_{t,s}k_{t,s}}$. Using this expression and $a^i_{t,s}=\alpha^i_{t,s}Q_{t,s}k_{t,s}$, one obtains:

$$\left(\omega^i_{t,s}-1\right)\left(r_{t,s}-r^f_{t,s}\right)-\omega^{i,s''}_{t,s} = \frac{\dot\alpha^i_{t,s}}{\alpha^i_{t,s}}.$$

In the expression above, substitute for $\omega^i_{t,s}-1$ using Eq. (13), which holds for the transition from s to s'. Then substitute for $r_{t,s}-r^f_{t,s}$ using the two-states portfolio-optimality

condition (IA.65), which holds with equality for the pessimists. Thereby, one obtains:

$$\left(1-\frac{\alpha^i_{t,s'}}{\alpha^i_{t,s}}\right)\left(\lambda^p_{s\to s''}\left(\frac{Q_{t,s'}}{Q_{t,s'}-Q_{t,s}}\frac{Q_{t,s''}-Q_{t,s}}{Q_{t,s''}}\right)+\lambda^p_{s\to s'}\frac{\alpha^p_{t,s}}{\alpha^p_{t,s'}}\right)-\omega^{i,s''}_{t,s}=\frac{\dot\alpha^i_{t,s}}{\alpha^i_{t,s}}. \tag{IA.70}$$

Using Eqs. (IA.60) and (IA.67), together with the expression $p^{s''}_{t,s}=\lambda^p_{s\to s''}Q_{t,s}/Q_{t,s''}$ derived above, we obtain, for each $i\in\{o,p\}$,

$$\omega^{i,s''}_{t,s}=\lambda^p_{s\to s''}(1-\omega^i_{t,s})\left(1-\frac{Q_{t,s}}{Q_{t,s''}}\right).$$

Using Eq. (13), this expression can equivalently be written as

$$\omega^{i,s''}_{t,s}=\lambda^p_{s\to s''}\left(1-\frac{\alpha^i_{t,s'}}{\alpha^i_{t,s}}\right)\frac{Q_{t,s'}}{Q_{t,s'}-Q_{t,s}}\left(1-\frac{Q_{t,s}}{Q_{t,s''}}\right). \tag{IA.71}$$

Substituting this expression into (IA.70), for $i\in\{o,p\}$, yields

$$\frac{\dot\alpha^i_{t,s}}{\alpha^i_{t,s}}=\lambda^p_{s\to s'}\frac{\alpha^p_{t,s}}{\alpha^p_{t,s'}}\left(1-\frac{\alpha^i_{t,s'}}{\alpha^i_{t,s}}\right). \tag{IA.72}$$

That is, Eq. (15), derived for the case of one successor state, also applies to the case of two successor states.

Wealth shares upon transition, when the leverage constraint does not bind. Suppose that optimists are unconstrained in state s. Then, Eq. (IA.65) holds with equality for $i\in\{o,p\}$. Subtracting Eq. (IA.65) for pessimists from the version for optimists, one obtains that $\lambda^o_{s\to s'}\frac{\alpha^o_{t,s}}{\alpha^o_{t,s'}}=\lambda^p_{s\to s'}\frac{\alpha^p_{t,s}}{\alpha^p_{t,s'}}$. Using this expression with $\alpha^o_{t,s}=1-\alpha^p_{t,s}$ and $\alpha^o_{t,s'}=1-\alpha^p_{t,s'}$, we obtain the first expression in Eq. (16). Using this result together with Eq. (15), we obtain the second expression in Eq. (16). Therefore, Eq. (16), derived for the case of one successor state, also applies to the case of two successor states.

E.2.2 Good booms, bad booms, and leaning

Here, we use Appendix E.2.1 to provide and prove the results discussed in Section 5.2 of the main text. From the boom state $s=1$, we allow for the possibility of transition to state $s''=1'$ with $\lambda^o_{1\to1'}=\lambda^p_{1\to1'}\equiv\lambda_{1\to1'}>0$. The new state is a non-speculative boom. It is a boom because $g_{1'}=g_1$. It is non-speculative because there is a leverage limit in state $s=1'$ given by $\bar\omega_{1'}=1$. Under belief type i, the non-speculative boom transitions to a bust with arrival rate $\lambda^i_{1'\to2}>0$, where $\lambda^i_{1'\to2}$ satisfies the same regularity conditions that $\lambda^i_{1\to2}$ does (see "Neutral rates" above). The Arrow-Debreu security is in zero net supply and has endogenous price $p^{1'}_{t,1}$ in $s=1$. This security pays 1 unit of the consumption

good if the boom turns non-speculative.

Monetary policy in the good boom. We focus on the case in which the central bank cannot distinguish between good and bad booms and hence must apply the same Q-targeting rule: $Q_1(\alpha) = Q_{1'}(\alpha)$. The following Lemma characterizes the implications of introducing the possibility of a good boom with $\lambda_{1\to 1'} > 0$.

Lemma IA.4 *The value of $\lambda_{1\to 1'} \geq 0$ has no effect on optimists' leverage $\omega_1^o(\alpha)$; the change in optimists' wealth share upon recession $\alpha_{1\to 2}(\alpha)$; and the growth rate of optimists' wealth share during the speculative boom $\dot{\alpha}_1(\alpha)$.*

Proof of Lemma IA.4. Suppose that the leverage constraint does not bind in $s = 1$. The post-bust optimist share $\alpha_{1\to 2}^{unc}(\alpha)$ is given by Eq. (16), hence it is independent of $\lambda_{1\to 1'}$. Using Eq. (IA.59), optimists' desired leverage, $\omega_1^{o,unc}(\alpha)$, satisfies:

$$\frac{\alpha_{1\to 2}^{unc}(\alpha)}{\alpha} = 1 - (\omega_1^{o,unc}(\alpha) - 1)\left(\frac{Q_1(\alpha)}{Q_2(\alpha_{1\to 2}^{unc}(\alpha))} - 1\right),$$

and therefore the equilibrium leverage $\omega_1^o(\alpha) = \min\{\omega_1^{o,unc}(\alpha), \bar{\omega}_1\}$ also does not depend on $\lambda_{1\to 1'}$. It then follows that using Eq. (IA.59), $\alpha_{1\to 2}(\alpha)$ is unchanged. From Eq. (IA.72), we have that $\dot{\alpha}_1(\alpha)$ is unchanged as well.

Next, we revisit a key result in the literature. In a setting with preannounced leaning, Caballero and Simsek (2020a) construct a class of leaning policies called "Prudential Monetary Policies" (PMP), which achieve the same macroprudential benefit (i.e., the same mapping from boom-time optimist wealth share to the optimist share upon recession) as with a regulatory tightening with leverage limit $\tilde{\omega} < \bar{\omega}$. We construct a version of PMP for our setting with unexpected leaning. We assume that, upon announcement, the planner follows a strategy of the form Eq. (35), replacing Q^{lull} with

$$Q_1(\alpha;\tilde{\omega}) = Q_2(\alpha_{1\to 2}(\alpha;\tilde{\omega}))\left(1 + \frac{\tilde{\omega} - 1}{\bar{\omega} - 1}\left(\frac{Q^*}{Q_2(\alpha_{1\to 2}(\alpha;\tilde{\omega}))} - 1\right)\right) \quad \text{for } \alpha > \bar{\boldsymbol{\alpha}}, \tag{IA.73}$$

where this rule is derived using Eq. (IA.59). That is, the planner initially sets the asset price equal to Q^* (i.e., $Q_1(\alpha;\tilde{\omega}) = Q^*$ for $\alpha \leq \bar{\boldsymbol{\alpha}}$) and thereafter transitions smoothly to the Q-targeting rule (IA.73). Conditional on α, (IA.73) achieves the same macroprudential benefit as tightening the leverage limit to $\tilde{\omega}$. Denote by $w_1^{pl}(\alpha, \bar{\omega}, Q_1(\cdot;\tilde{\omega}))$ the gap value when the leverage limit is $\bar{\omega}$ and the planner follows a rule of the form (35) that transitions

smoothly to $Q_1(\cdot;\tilde{\omega})$. Denote by $w_1^{pl}(\alpha;\tilde{\omega})$ the gap value when the planner follows a Wicksellian rule, with leverage limit $\tilde{\omega}$. The next result shows that, with this version of PMP, a key result in Caballero and Simsek (2020a) holds in our setting with $\lambda_{1\to 1'} \geq 0$. The proof follows closely Caballero and Simsek (2020a), using the results thus far in our Appendix E.2 to show that the welfare result remains when taking into account the possibility of good booms.

Lemma IA.5 *Suppose $\lambda_{1\to 1'} \geq 0$. For small policy changes, the welfare effects of phased-in PMP are the same as the welfare effects of tightening the leverage limit directly:*

$$\lim_{\hat{\alpha}\to 0} \frac{dw_1^{pl}(\alpha,\bar{\omega},Q_1(\cdot;\tilde{\omega}))}{d\tilde{\omega}}\Big|_{\tilde{\omega}=\bar{\omega}} = \frac{dw_1^{pl}(\alpha;\tilde{\omega})}{d\tilde{\omega}}\Big|_{\tilde{\omega}=\bar{\omega}}. \tag{IA.74}$$

Proof of Lemma IA.5. With $\lambda_{1\to 1'} \geq 0$ and leverage limit $\bar{\omega}$, with a smoothed transition to Q-targeting rule $Q_1(\alpha;\tilde{\omega})$, the gap value in state 1 at time τ_1 is[17]

$$w_1^{pl}(\alpha_{\tau_1,1},\bar{\omega},Q_1(\cdot;\tilde{\omega})) = \int_{\tau_1}^{\infty} e^{-(\rho+\lambda_{1\to 2}^{pl}+\lambda_{1\to 1'}^{pl})(t-\tau_1)} \left(W(Q_1(\alpha_{t,1};\tilde{\omega})) + \lambda_{1\to 2}^{pl} w_{t,2}^{pl}(\alpha_{1\to 2}(\alpha_{t,1})) + \lambda_{1\to 1'}^{pl} w_{t,1'}^{pl}(\alpha_{1\to 1'}(\alpha_{t,1}),Q_1(\cdot;\tilde{\omega}))\right) dt, \tag{IA.75}$$

with $\alpha_{1\to 1'}(\alpha) = \alpha$ (as shown in Appendix E.2.1), and, if the leverage constraint is binding,

$$\alpha_{1\to 2}(\alpha) = \alpha\left[1-(\bar{\omega}-1)\left(\frac{Q_1(\alpha;\tilde{\omega})}{Q_2(\alpha_{1\to 2}(\alpha))}-1\right)\right].$$

Under Q-targeting rule $Q_1(\alpha;\tilde{\omega})$ the gap value in state $s=1'$ at time $\tau_{1'}$ is:

$$w_{1'}^{pl}(\alpha_{\tau_{1'},1'},Q_1(\cdot;\tilde{\omega})) = \int_{\tau_{1'}}^{\infty} e^{-(\rho+\lambda_{1'\to 2}^{pl})(t-\tau_{1'})} \left(W(Q_{1'}(\alpha_{t,1'};\tilde{\omega})) + \lambda_{1'\to 2}^{pl} w_{t,2}^{pl}(\alpha_{1'\to 2}(\alpha_{t,1'}))\right) dt, \tag{IA.76}$$

with $Q_{1'}(\alpha) = Q_1(\alpha)$ (the central bank applies the same Q-targeting rule in each state), and $\alpha_{1'\to 2}(\alpha) = \alpha$ (state $s=1'$ is a non-speculative boom with no leverage).

With Wicksellian policy and leverage limit $\tilde{\omega} < \bar{\omega}$, the gap value at τ_1 in state $s=1$ is:

$$w_1^{pl}(\alpha_{\tau_1,1};\tilde{\omega}) = \int_{\tau_1}^{\infty} e^{-(\rho+\lambda_{1\to 2}^{pl}+\lambda_{1\to 1'}^{pl})(t-\tau_1)} \left(W(Q^*) + \lambda_{1\to 2}^{pl} w_{t,2}^{pl}(\alpha_{1\to 2}(\alpha_{t,1};\tilde{\omega})) + \lambda_{1\to 1'}^{pl} w_{t,1'}^{pl}(\alpha_{1\to 1'}(\alpha_{t,1}),Q^*)\right) dt, \tag{IA.77}$$

[17] Regarding the limit as $\hat{\alpha}\to 0$, the proof relies on continuous dependence as in earlier proofs.

with, if the leverage constraint is binding,

$$\alpha_{1\to 2}(\alpha;\tilde{\omega}) = \alpha\left[1-(\tilde{\omega}-1)\left(\frac{Q^*}{Q_2(\alpha_{1\to 2}(\alpha;\tilde{\omega}))}-1\right)\right].$$

With Wicksellian policy and leverage limit $\tilde{\omega} < \bar{\omega}$, the gap value at $\tau_{1'}$ in state $s = 1'$ is:

$$w^{pl}_{1'}(\alpha_{\tau_{1'},1'}, Q^*) = \int_{\tau_{1'}}^{\infty} e^{-(\rho+\lambda^{pl}_{1'\to 2})(t-\tau_{1'})}\left(W(Q^*)+\lambda^{pl}_{1'\to 2} w^{pl}_{t,2}(\alpha_{1'\to 2}(\alpha_{t,1'}))\right)dt.$$

Subtracting (IA.77) from (IA.75),

$$\begin{aligned} w^{pl}_1(\alpha_{\tau_1,1},\bar{\omega},Q_1(\cdot;\tilde{\omega})) - w^{pl}_1(\alpha_{\tau_1,1};\tilde{\omega}) = \\ \int_{\tau_1}^{\infty} e^{-(\rho+\lambda^{pl}_{1\to 2}+\lambda^{pl}_{1\to 1'})(t-\tau_1)}\Big(W(Q_1(\alpha_{t,1};\tilde{\omega})) - W(Q^*) + \\ \lambda^{pl}_{1\to 1'}(w^{pl}_{t,1'}(\alpha_{1\to 1'}(\alpha_{t,1}),Q_1(\cdot;\tilde{\omega})) - w^{pl}_{t,1'}(\alpha_{1\to 1'}(\alpha_{t,1}),Q^*))\Big)dt. \end{aligned} \quad \text{(IA.78)}$$

with $\alpha_{1\to 1'}(\alpha_{t,1}) = \alpha_{t,1}$. Similarly,

$$w^{pl}_{1'}(\alpha_{\tau_{1'},1'},Q_1(\cdot;\tilde{\omega})) - w^{pl}_{1'}(\alpha_{\tau_{1'},1'},Q^*) = \int_{\tau_{1'}}^{\infty} e^{-(\rho+\lambda^{pl}_{1'\to 2})(t-\tau_{1'})}\left(W(Q_{1'}(\alpha_{t,1'};\tilde{\omega})) - W(Q^*)\right)dt. \quad \text{(IA.79)}$$

These expressions obtain because, by construction, for $s \in \{1,1'\}$, the evolution of optimists' wealth share upon transition to $s = 2$ is the same under: (i) the Q-targeting rule in Eq. (IA.73) with leverage limit $\bar{\omega}$; and (ii) a Wicksellian policy with $Q_1(\alpha) = Q^*$ and leverage limit $\tilde{\omega}$. That is, $\alpha_{1\to 2}(\alpha) = \alpha_{1\to 2}(\alpha;\tilde{\omega})$ and $\alpha_{1'\to 2}(\alpha) = \alpha_{1'\to 2}(\alpha;\tilde{\omega}) = \alpha$. Hence, from Eq. (IA.72), optimists' wealth share growth absent state transition is also the same under these two policy scenarios: $\dot{\alpha}_1(\alpha) = \dot{\alpha}_1(\alpha;\tilde{\omega})$. Similarly, in state $s = 1'$, optimists' wealth share growth and optimists' wealth share upon recession are the same in the two scenarios. Then, differentiating Eqs. (IA.78)–(IA.79) with respect to $\tilde{\omega}$ and evaluating at $\tilde{\omega} = \bar{\omega}$, and using $\frac{dW(Q_1)}{dQ_1}|_{Q_1=Q^*} = 0$, we obtain Eq. (IA.74).

E.3 The anticipation effect

This appendix provides results and proofs described in Section 5.3 of the main text. We assume that (g_3, g_4, g_5) and $(\lambda^i_{3\to 4}, \lambda^i_{4\to 5}), \forall i$ satisfy the same parametric conditions as (g_1, g_2, g_3) and $(\lambda^i_{1\to 2}, \lambda^i_{2\to 3})$ do in the benchmark model. In the quantitative analysis, we further assume that $g_s = g_{s+2}$ for $s \in \{1,2,3\}$ and that $\lambda^i_{s\to s+1} = \lambda^i_{s+2\to s+3}$ for $s \in \{1,2\}$. That is, in the quantitative analyses of the two-cycle model, the two booms are alike (and the two busts are alike) in the growth rate of capital and transition rates. The admissible

policies are continuous and right-differentiable in their arguments.

The next result, Lemma IA.6, compares outcomes during the recession $s = 2$ while varying the degree of leaning during the $s = 3$ boom. We say that a Q-targeting rule $Q_3^{aggr}(\alpha; \underline{\boldsymbol{\alpha}})$ is more aggressive than $Q_3^{mild}(\alpha; \underline{\boldsymbol{\alpha}})$ if $Q_3^{aggr}(\alpha; \underline{\boldsymbol{\alpha}}) \leq Q_3^{mild}(\alpha; \underline{\boldsymbol{\alpha}})$ for all $(\alpha, \underline{\boldsymbol{\alpha}}) \in [0,1] \times [0,1]$, with strict inequality for $(\alpha, \underline{\boldsymbol{\alpha}}) \in (0,1) \times (0,1)$. This is a partial ordering of Q-targeting rules.

Lemma IA.6 (Anticipation effects) *When investors anticipate more aggressive leaning in the $s = 3$ boom: the asset price $Q_2(\alpha)$ and utilization $\eta_2(\alpha)$ are strictly lower, for all $\alpha \in (0,1)$; and, conditional on the optimist wealth share $\alpha_{\tau_2,2}$ at the start of the first bust, the time-paths for the asset price and utilization, $\{Q_{\tau,2}, \eta_{\tau,2}\}_{\tau > \tau_2}$, are strictly lower.*

An intuition for the proof can be explained for the case in which, in the all-pessimist economy, the degree of leaning, if any, is the same under the two rules: $Q_3^{aggr}(0,0) = Q_3^{mild}(0,0)$. For small but strictly positive values of the optimist wealth share, more aggressive policy reduces the elasticity of the recessionary asset price to the optimist wealth share, $d \ln Q_2(\alpha)/d\alpha$. Hence, as the $s = 3$ leaning policy becomes more aggressive, the recessionary asset price for a given optimist wealth share has to fall, in order to guarantee that it converges to the asset price in the all-pessimist economy (as $\alpha \to 0$).

Proof of Lemma IA.6. Suppose $Q_3^{aggr}(\cdot)$ is more aggressive than $Q_3^{mild}(\cdot)$.

With rule $Q_3(\alpha, \underline{\boldsymbol{\alpha}})$, the $s = 2$ asset price with all pessimists is $Q_2^p = Q_2(0)$, with

$$\rho + g_2 - \delta \left(\frac{Q_2^p}{Q^*} \eta^* \right) + \lambda_{2\to 3}^p \left(1 - \frac{Q_2^p}{Q_3^p} \right) = 0, \tag{IA.80}$$

where $Q_3^p = Q_3(0;0)$. The left side of (IA.80) is decreasing in Q_2^p and increasing in Q_3^p. Hence, $Q_2^{aggr,p} \leq Q_2^{mild,p}$. Following very similar steps, one obtains $Q_2^{aggr,o} \leq Q_2^{mild,o}$.

We now show $Q_2^{aggr}(\alpha) < Q_2^{mild}(\alpha)$ for $\alpha \in (0,1)$. Define $f(\alpha) = \ln Q_2^{aggr}(\alpha) - \ln Q_2^{mild}(\alpha)$. By the extreme value theorem, this expression has a minimum and a maximum on the interval $[0,1]$. As shown earlier in this proof, $Q_2^{aggr,i} \leq Q_2^{mild,i}$ for $i \in \{o,p\}$. Therefore, $f(\alpha) \leq 0$ for $\alpha \in \{0,1\}$. Using $\dot{\alpha}_2(\alpha) = \alpha(1-\alpha)(\lambda_{2\to 3}^p - \lambda_{2\to 3}^o) < 0$ ((16) applies) and (40),

$$f'(\alpha) = -\dot{\alpha}_2(\alpha)^{-1} \left(\delta \left(\frac{Q_2^{mild}(\alpha)}{Q^*} \eta^* \right) - \delta \left(\frac{Q_2^{aggr}(\alpha)}{Q^*} \eta^* \right) + \bar{\lambda}_{2\to 3}(\alpha) \left(\frac{Q_2^{mild}(\alpha)}{\underline{Q}_3^{mild}(\alpha)} - \frac{Q_2^{aggr}(\alpha)}{\underline{Q}_3^{aggr}(\alpha)} \right) \right), \tag{IA.81}$$

where, following the earlier notation, $\underline{Q}_3(\alpha) = Q_3(\alpha_{2\to3}(\alpha);\alpha)$. Consider a critical point $\alpha \in (0,1)$ with $f'(\alpha) = 0$. Using Eq. (IA.81), we have that at such a critical point,

$$\left(\delta\left(\frac{Q_2^{mild}(\alpha)}{Q^*}\eta^*\right) - \delta\left(\frac{Q_2^{aggr}(\alpha)}{Q^*}\eta^*\right) + \bar{\lambda}_{2\to3}(\alpha)\left(\frac{Q_2^{mild}(\alpha)}{\underline{Q}_3^{mild}(\alpha)} - \frac{Q_2^{aggr}(\alpha)}{\underline{Q}_3^{aggr}(\alpha)}\right)\right) = 0. \quad \text{(IA.82)}$$

Note that the left side of Eq. (IA.82) is increasing in $Q_2^{mild}(\alpha)$ and decreasing in $Q_2^{aggr}(\alpha)$, with $\underline{Q}_3^{aggr}(\alpha) < \underline{Q}_3^{mild}(\alpha)$. Hence, at such a critical point, $Q_2^{aggr}(\alpha) < Q_2^{mild}(\alpha)$. Thus, $f(\alpha) < 0$ for $\alpha \in (0,1)$. Also, the maximum of $f(\alpha)$ on $[0,1]$ is weakly negative.

The lemma's second part then obtains because, in the absence of a leverage limit during $s = 2$, the path for the optimist wealth share $\alpha_{\tau,2}$ for $\tau > \tau_2$ is determined by the initial condition $\alpha_{\tau_2,2}$ and $\dot{\alpha}_2(\alpha) = \alpha(1-\alpha)(\lambda^p_{2\to3} - \lambda^o_{2\to3}) < 0$, which do not depend on the anticipated monetary policy in the $s = 3$ boom.

Planner's problem with the anticipation effect. Denote the economy's initial condition, the optimists' wealth share at $t = 0$, by $\alpha_{t=0}$. The planner's problem is:[18]

$$\sup_{\{Q_1(\alpha)\in\mathcal{C}_r, Q_3(\alpha;\underline{\boldsymbol{\alpha}})\in\mathcal{C}_r\}} w_1^{pl}(\alpha_{t=0})$$

subject to the risk-balance condition (40), the law of motion for optimists' wealth share upon transition from $s = 3$ to $s = 4$,

$$\frac{\alpha_{3\to4}(\alpha;\underline{\boldsymbol{\alpha}})}{\alpha} - 1 = (\omega_3^o(\alpha;\underline{\boldsymbol{\alpha}}) - 1)\left(1 - \frac{Q_3(\alpha;\underline{\boldsymbol{\alpha}})}{Q_4(\alpha_{3\to4}(\alpha;\underline{\boldsymbol{\alpha}}))}\right),$$

and the evolution of optimists' wealth share absent transition in $s = 3$,

$$\frac{\dot{\alpha}_3(\alpha;\underline{\boldsymbol{\alpha}})}{\alpha} = \lambda^p_{3\to4}\frac{1-\alpha}{1-\alpha_{3\to4}(\alpha;\underline{\boldsymbol{\alpha}})}\left(1 - \frac{\alpha_{3\to4}(\alpha;\underline{\boldsymbol{\alpha}})}{\alpha}\right),$$

where, for $s \in \{1,2,4\}$, optimists' wealth share growth and change in wealth share upon transition to $s' = s+1$ are given by Eqs. (13) and (15). In state $s = 4$, the asset price satisfies:[19]

$$\rho + g_4 - \delta\left(\frac{Q_4(\alpha)}{Q^*}\eta^*\right) - \frac{Q_4'(\alpha)}{Q_4(\alpha)}\alpha(1-\alpha)(\lambda^o_{4\to5} - \lambda^p_{4\to5}) + \bar{\lambda}_{4\to5}(\alpha)\left(1 - \frac{Q_4(\alpha)}{Q^*}\right) = 0. \quad \text{(IA.83)}$$

Here, we allow the Q-targeting rule to differ across booms $s \in \{1,3\}$. (In the infinitely recurrent version of the model, there is a single rule applied across all booms.)

The next result shows that higher anticipated asset prices at the start of the $s = 3$

[18] We write $Q_3(\alpha;\underline{\boldsymbol{\alpha}}) \in \mathcal{C}_r$ to mean Q_3 is jointly continuous on $[0,1]^2$ and right-differentiable in each argument.

[19] The boundary condition is given by (IA.83) for $\alpha = 0$.

boom weakly raise the asset price in the preceding recession.

Lemma IA.7 *Suppose that:*

(i) $Q_3(\alpha_{2\to3}(\underline{\boldsymbol{\alpha}});\underline{\boldsymbol{\alpha}}) \leq \hat{Q}_3(\alpha_{2\to3}(\underline{\boldsymbol{\alpha}});\underline{\boldsymbol{\alpha}})$ *for* $\underline{\boldsymbol{\alpha}} \in [0,1]$*; and*

(ii) there exists a pair $\{\alpha',\alpha''\}$ *such that* $Q_3(\alpha_{2\to3}(\underline{\boldsymbol{\alpha}});\underline{\boldsymbol{\alpha}}) < \hat{Q}_3(\alpha_{2\to3}(\underline{\boldsymbol{\alpha}});\underline{\boldsymbol{\alpha}})$ *for* $\underline{\boldsymbol{\alpha}} \in (\alpha',\alpha'')$*, with* $0 \leq \alpha' < \alpha'' \leq 1$.

Then, $Q_2(\alpha) < \hat{Q}_2(\alpha)$ *for* $\alpha \in (\alpha',\alpha'')$ *and* $Q_2(\alpha) \leq \hat{Q}_2(\alpha)$ *for* $\alpha \in [0,1]$.

Proof of Lemma IA.7. As a starting point, note from the proof of Lemma IA.6 that $Q_2^p \leq \hat{Q}_2^p$ under assumption (i) in Lemma IA.7. We next show that $Q_2(\alpha') \leq \hat{Q}_2(\alpha')$. We offer a proof by contradiction. Given that $Q_2^p \leq \hat{Q}_2^p$, there are only two ways that $Q_2(\alpha') > \hat{Q}_2(\alpha')$ might obtain. First, $Q_2(\alpha) > \hat{Q}_2(\alpha), \forall \alpha \in (0,\alpha']$. Second, there exists $\tilde{\alpha} \in (0,\alpha')$ with $Q_2(\tilde{\alpha}) = \hat{Q}_2(\tilde{\alpha})$ and $\frac{dQ_2(\alpha)}{d\alpha}\big|_{\alpha=\tilde{\alpha}} > \frac{d\hat{Q}_2(\alpha)}{d\alpha}\big|_{\alpha=\tilde{\alpha}}$. We will show that neither of these two possibilities can obtain under the assumptions in the Lemma.

Suppose first that $Q_2(\alpha) > \hat{Q}_2(\alpha), \forall \alpha \in (0,\alpha']$. From Eq. (IA.81), this assumption implies that $\frac{d\ln Q_2(\alpha)}{d\alpha} - \frac{d\ln \hat{Q}_2(\alpha)}{d\alpha} < 0$ for $\alpha \in (0,\alpha')$, which contradicts $Q_2(\alpha) > \hat{Q}_2(\alpha)$ for $\alpha \in (0,\alpha']$ (recall that $Q_2^p \leq \hat{Q}_2^p$). Second, suppose instead that there exists $\tilde{\alpha} \in (0,\alpha')$ with $Q_2(\tilde{\alpha}) = \hat{Q}_2(\tilde{\alpha})$ and $\frac{dQ_2(\alpha)}{d\alpha}\big|_{\alpha=\tilde{\alpha}} > \frac{d\hat{Q}_2(\alpha)}{d\alpha}\big|_{\alpha=\tilde{\alpha}}$. From Eq. (IA.81) and assumption (i) of the Lemma, $Q_2(\tilde{\alpha}) = \hat{Q}_2(\tilde{\alpha})$ implies $\frac{d\ln Q_2(\alpha)}{d\alpha} - \frac{d\ln \hat{Q}_2(\alpha)}{d\alpha} \leq 0$, a contradiction. By a very similar argument, one can show that $Q_2(\alpha'') \leq \hat{Q}_2(\alpha'')$. The remainder of the argument follows the proof of Lemma IA.6. $\ln(Q_2(\alpha)) - \ln(\hat{Q}_2(\alpha)) \leq 0$ for $\alpha \in \{\alpha',\alpha''\}$. Then, from Eq. (IA.81) and assumption (ii), $\ln(Q_2(\alpha)) - \ln(\hat{Q}_2(\alpha)) < 0$ for $\alpha \in (\alpha',\alpha'')$.

The next result shows that, in searching for optimal $s = 3$ rules, it is sufficient to restrict attention to phased-in rules that set the asset price equal to Q^* at the start of $s = 3$. The argument uses a family of auxiliary conditional planner's problems. Fix an admissible Q_3 rule and, for $\theta \in [0,1]$, define

$$Q_{3,\theta}(\alpha;\underline{\boldsymbol{\alpha}}) \equiv (1-\theta)Q_3(\alpha;\underline{\boldsymbol{\alpha}}) + \theta Q^*. \tag{IA.84}$$

Let $Q_{2,\theta}(\cdot)$ be the recession asset price induced by the boom-arrival values of $Q_{3,\theta}$. Thus, $\theta = 0$ corresponds to the original anticipated rule, whereas $\theta = 1$ sets the boom-arrival price equal to Q^*. By Lemma IA.7, $Q_{2,\theta}(\alpha)$ is weakly increasing in θ.

For purposes of comparing the conditional problems, hold fixed the continuation

value generated by the original Q_3 rule and define

$$w^{pl}_{2,\theta}(\alpha) \equiv \int_0^\infty e^{-(\rho+\lambda^{pl}_{2\to3})t}\left[W(Q_{2,\theta}(\alpha_{t,2})) + \lambda^{pl}_{2\to3} w^{pl}_3(\alpha_{2\to3}(\alpha_{t,2});\alpha_{t,2})\right] dt. \tag{IA.85}$$

The $s=2$ path $\{\alpha_{t,2}\}$ is independent of the anticipated Q-targeting rule. Since $Q_{2,\theta} \leq Q^*$ and W is increasing below Q^*, $w^{pl}_{2,\theta}(\alpha)$ is also weakly increasing in θ.

For each θ, consider the conditional planner's problem of choosing Q_1, given $\left(Q_{2,\theta}, w^{pl}_{2,\theta}\right)$. In these auxiliary problems only, temporarily relax the upper bound $Q_1 \leq Q^*$ and assume that firms meet the associated demand if $Q_1 > Q^*$. All other requirements on the rule are maintained. We assume that an optimal rule $Q^{\text{opt}}_{1,\theta}$ can be chosen such that

$$Q^{\text{opt}}_{1,\theta}(\alpha) \leq Q^*, \qquad \text{for almost every } \alpha \in [0,1] \text{ and every } \theta \in [0,1], \tag{IA.86}$$

with corresponding gap value $w^{pl,\text{opt}}_{1,\theta}(\alpha)$. In words, relaxing the upper bound on the boom asset price does not make it optimal for the planner to generate overutilization. This is a sufficient condition and can be weakened. For example, some overutilization could be permitted if it is small, since $W'(Q^*)=0$, and its welfare cost is outweighed by the improvement in the $s=2$ gap value. More generally, overutilization losses could be outweighed by utilization gains at wealth shares where $Q^{\text{opt}}_{1,\theta} < Q^*$.

One sufficient condition for Eq. (IA.86) is

$$Q^{\text{bind}}_{1,\theta}(\alpha) < Q^*, \qquad \forall(\theta,\alpha) \text{ relevant to the auxiliary conditional problems}, \tag{IA.87}$$

where $Q^{\text{bind}}_{1,\theta}(\alpha)$ is the threshold below which optimists' leverage constraint binds. This condition does not require the constraint to be slack at the lower asset prices associated with leaning. It requires slackness only at asset prices at or above Q^*. At those prices, Eq. (16) implies that the wealth-share mappings are independent of Q_1. Raising Q_1 above Q^* then leaves optimists' wealth-share growth unaffected and lowers $W(Q_1)$.[20]

Let $\mathcal{A}$ denote the set of admissible policy pairs (Q_1, Q_3): $Q_1 \in \mathcal{C}_r$ and $Q_3(\alpha;\underline{\boldsymbol{\alpha}}) \in \mathcal{C}_r$, with both rules bounded above by Q^* and satisfying the cure-no-worse-than-the-disease property. Let

$$\mathcal{A}^0 \equiv \left\{(Q_1,Q_3) \in \mathcal{A} : Q_3(\alpha_{2\to3}(\underline{\boldsymbol{\alpha}});\underline{\boldsymbol{\alpha}}) = Q^* \text{ for every } \underline{\boldsymbol{\alpha}} \in [0,1]\right\}.$$

[20] Let $a_\theta(\alpha,Q) \equiv \alpha_{1\to2,\theta}(\alpha;Q)$. Another sufficient condition is $\lambda^{pl}_{1\to2}\frac{\partial w^{pl}_{2,\theta}(a)}{\partial a} \geq \lambda^{p}_{1\to2}\frac{\partial w^{pl}_{1,\theta}(\alpha)}{\partial \alpha}\left(\frac{1-\alpha}{1-a}\right)^2$, for $a = a_\theta(\alpha,Q)$ and $Q \geq Q^*$, whenever optimists' leverage constraint binds.

Lemma IA.8 (No leaning at boom arrival) *For every initial condition* $\alpha_{t=0} \in (0,1)$*,*

$$\sup_{(Q_1,Q_3)\in\mathcal{A}} w_1^{pl}(\alpha_{t=0}; Q_1, Q_3) = \sup_{(Q_1,Q_3)\in\mathcal{A}^0} w_1^{pl}(\alpha_{t=0}; Q_1, Q_3).$$

Equivalently, for every $(Q_1,Q_3) \in \mathcal{A}$ *and every* $\varepsilon > 0$*, there exists* $(Q_1^\varepsilon, Q_3^\varepsilon) \in \mathcal{A}^0$ *such that*

$$w_1^{pl}(\alpha_{t=0}; Q_1^\varepsilon, Q_3^\varepsilon) \geq w_1^{pl}(\alpha_{t=0}; Q_1, Q_3) - \varepsilon.$$

Proof of Lemma IA.8. Since $\mathcal{A}^0 \subseteq \mathcal{A}$, it suffices to prove the reverse inequality. Fix $(Q_1,Q_3) \in \mathcal{A}$ and construct the auxiliary problems above. At $\theta = 0$, the original rule Q_1 is feasible and $w_{2,0}^{pl}$ is the original $s = 2$ gap value. Hence,

$$w_{1,0}^{pl,\mathrm{opt}}(\alpha_{t=0}) \geq w_1^{pl}(\alpha_{t=0}; Q_1, Q_3). \tag{IA.88}$$

We next show that the optimized conditional gap value is weakly increasing in θ. Fix $\theta < 1$, let

$$a_\theta(\alpha) \equiv \alpha_{1\to 2,\theta}\big(\alpha; Q_{1,\theta}^{\mathrm{opt}}(\alpha)\big),$$

and, for sufficiently small $\zeta > 0$, consider

$$Q_{1,\theta}^{\zeta}(\alpha) \equiv Q_{1,\theta}^{\mathrm{opt}}(\alpha)\frac{Q_{2,\theta+\zeta}(a_\theta(\alpha))}{Q_{2,\theta}(a_\theta(\alpha))}. \tag{IA.89}$$

By Eqs. (13) and (15), this scaling leaves $a_\theta(\alpha)$ and $\dot{\alpha}_{1,\theta}(\alpha)$ unchanged. Thus, when $Q_{1,\theta}^{\zeta}$ is used in the $(\theta+\zeta)$ conditional problem, the $s = 1$ path of optimists' wealth share is unchanged.

Conditional optimality and comparison along this common path imply

$$D_+ w_{1,\theta}^{pl,\mathrm{opt}}(\alpha_{t=0}) \geq \int_0^\infty e^{-(\rho+\lambda_{1\to 2}^{pl})t}\Bigg[W'\big(Q_{1,\theta}^{\mathrm{opt}}(\alpha_{t,1}^\theta)\big)\, Q_{1,\theta}^{\mathrm{opt}}(\alpha_{t,1}^\theta)\frac{\partial}{\partial\theta}\log Q_{2,\theta}\big(a_\theta(\alpha_{t,1}^\theta)\big) + \lambda_{1\to 2}^{pl}\frac{\partial w_{2,\theta}^{pl}}{\partial\theta}\left(a_\theta(\alpha_{t,1}^\theta)\right)\Bigg]dt \geq 0,$$

where D_+ denotes the lower right derivative and $\{\alpha_{t,1}^\theta\}_{t\geq 0}$ is the path generated by $Q_{1,\theta}^{\mathrm{opt}}$. The final inequality follows from Eq. (IA.86), $W'(Q) \geq 0$ for $Q \leq Q^*$, and the monotonicity of $Q_{2,\theta}$ and $w_{2,\theta}^{pl}$ in θ. At points where $Q_{1,\theta}^{\mathrm{opt}} = Q^*$, any overshoot under Eq. (IA.89) has only an $o(\zeta)$ welfare cost because $W'(Q^*) = 0$. It follows that

$$w_{1,1}^{pl,\mathrm{opt}}(\alpha_{t=0}) \geq w_{1,0}^{pl,\mathrm{opt}}(\alpha_{t=0}).$$

The next step is to approximate the hybrid problem at $\theta = 1$ using admissible $s = 3$ rules. Let $\varphi : \mathbb{R} \to [0,1]$ be continuous, right-differentiable, and weakly decreasing, with

$\varphi(z) = 1$ for $z \leq 0$ and $\varphi(z) = 0$ for $z \geq 1$. For $\hat{\alpha} > 0$, write $\alpha' = \alpha_{2\to 3}(\underline{\boldsymbol{\alpha}})$ and define

$$Q_3^{\hat{\alpha}}(\alpha; \underline{\boldsymbol{\alpha}}) \equiv \varphi\left(\frac{\alpha - \alpha'}{\hat{\alpha}}\right) Q^* + \left[1 - \varphi\left(\frac{\alpha - \alpha'}{\hat{\alpha}}\right)\right] Q_3(\alpha; \underline{\boldsymbol{\alpha}}). \tag{IA.90}$$

Then $Q_3 \leq Q_3^{\hat{\alpha}} \leq Q^*$, $Q_3^{\hat{\alpha}}(\alpha'; \underline{\boldsymbol{\alpha}}) = Q^*$, and $Q_3^{\hat{\alpha}}$ equals the original rule once $\alpha \geq \alpha' + \hat{\alpha}$. Because the auxiliary optimizers belong to $\mathcal{C}_r$, Eq. (IA.86) implies the upper bound everywhere. Hence, $(Q_{1,1}^{\text{opt}}, Q_3^{\hat{\alpha}}) \in \mathcal{A}^0$.

Every such pair induces $Q_{2,1}$, because Eq. (40) depends on the $s = 3$ rule only through its boom-arrival value. Moreover, $Q_3^{\hat{\alpha}}$ differs from the original rule along the $s = 3$ path only over a wealth-share interval whose length converges to zero. The same phase-in and dominated-convergence argument as in the proof of Proposition 2 therefore gives

$$w_1^{pl}\left(\alpha_{t=0}; Q_{1,1}^{\text{opt}}, Q_3^{\hat{\alpha}}\right) \to w_{1,1}^{pl,\text{opt}}(\alpha_{t=0}) \qquad \text{as } \hat{\alpha} \to 0.$$

Together with Eq. (IA.88), this implies that, for every $\varepsilon > 0$, some $(Q_{1,1}^{\text{opt}}, Q_3^{\hat{\alpha}}) \in \mathcal{A}^0$ satisfies

$$w_1^{pl}\left(\alpha_{t=0}; Q_{1,1}^{\text{opt}}, Q_3^{\hat{\alpha}}\right) \geq w_1^{pl}(\alpha_{t=0}; Q_1, Q_3) - \varepsilon.$$

Taking the supremum over $(Q_1, Q_3) \in \mathcal{A}$ proves the result.

Proof of Proposition 5. Let $\alpha' = \alpha_{2\to 3}(\underline{\boldsymbol{\alpha}})$. By Lemma IA.8, the supremum over admissible (Q_1, Q_3) is unchanged if we restrict attention to Q_3 rules that satisfy the boom-arrival restriction $Q_3(\alpha'; \underline{\boldsymbol{\alpha}}) = Q^*, \forall \underline{\boldsymbol{\alpha}} \in [0,1]$. After the boom-arrival time (i.e., beyond τ_3), the planner faces exactly the same trade-off as in the single-cycle boom problem (with $S \equiv \{1,2,3\}$ and the state variable initialized at α'). Hence, by definition of $Q^{\text{lull}}(\cdot)$, the supremum is consistent with $Q_3(\alpha') = Q^*$ and $Q_3(\alpha) = Q^{\text{lull}}(\alpha)$ for $\alpha > \alpha'$. We refer to this rule as $Q_3^{(0)}(\cdot; \underline{\boldsymbol{\alpha}})$; the superscript 0 refers to an instantaneous transition to $Q^{\text{lull}}(\cdot)$ after the first instant of the boom. When $Q^{\text{lull}}(\alpha') = Q^*$, the rule $Q_3^{(0)}(\cdot; \underline{\boldsymbol{\alpha}})$ is admissible. This proves part (i) of the Proposition. When $Q^{\text{lull}}(\alpha') < Q^*$, however, $Q_3^{(0)}(\cdot; \underline{\boldsymbol{\alpha}})$ has a discontinuity at α' and is not $\mathcal{C}_r$. However, the family of rules $Q_3(\alpha; \underline{\boldsymbol{\alpha}}, \hat{\alpha})$ described in Proposition 5 are admissible for $\hat{\alpha} > 0$. Moreover, $Q_3(\cdot; \underline{\boldsymbol{\alpha}}, \hat{\alpha})$ converges pointwise to $Q_3^{(0)}(\cdot; \underline{\boldsymbol{\alpha}})$ as $\hat{\alpha} \to 0$.

Next, let $Q_2^{(\hat{\alpha})}$ denote the solution to (40) under $Q_3(\cdot; \underline{\boldsymbol{\alpha}}, \hat{\alpha})$, and Q_2^0 the solution under $Q_3^{(0)}(\cdot; \underline{\boldsymbol{\alpha}})$. By (42), both rules satisfy $Q_3(\alpha_{2\to 3}(\underline{\boldsymbol{\alpha}}); \underline{\boldsymbol{\alpha}}, \hat{\alpha}) = Q_3^{(0)}(\alpha_{2\to 3}(\underline{\boldsymbol{\alpha}}); \underline{\boldsymbol{\alpha}}) = Q^*, \forall \underline{\boldsymbol{\alpha}} \in [0,1]$. Hence, in (40) the boom-state valuation term $Q_3(\alpha_{2\to 3}(\alpha); \alpha)$ equals Q^* pointwise under both rules. Thus, $Q_2^{(\hat{\alpha})}$ and Q_2^0 satisfy the same ODE and the same boundary

condition, and therefore coincide: $Q_2^{(\hat{\alpha})}(\alpha) = Q_2^0(\alpha)$ for all $\alpha \in [0,1]$. Hence the induced paths of utilization in $s=2$ are the same. Although $Q_3(\cdot;\underline{\boldsymbol{\alpha}},\hat{\alpha})$ and $Q_3^{(0)}(\cdot;\underline{\boldsymbol{\alpha}})$ generally differ for $\alpha > \alpha_{2\to3}(\underline{\boldsymbol{\alpha}})$, these differences affect outcomes only *after* the transition to $s=3$, whereas (40) depends on the $s=3$ rule only through the boom-arrival price.

Then, consider the $s=1$ rule $Q_1^{(\hat{\alpha})}$ that maximizes the $s=1$ gap value, given $Q_3(\cdot;\underline{\boldsymbol{\alpha}},\hat{\alpha})$. By dominated convergence (see Proof of Proposition 2), the $s=1$ gap value satisfies

$$w_1^{pl}\big(\alpha_{t=0};\, Q_1^{(\hat{\alpha})}, Q_3(\cdot;\underline{\boldsymbol{\alpha}},\hat{\alpha})\big) \;\to\; \sup_{Q_1\in\mathcal{C}_r} w_1^{pl}\big(\alpha_{t=0};\, Q_1, Q_3^0(\cdot;\underline{\boldsymbol{\alpha}})\big) \text{ as } \hat{\alpha}\to 0. \tag{IA.91}$$

The right-hand side equals the supremum of the planner's problem over all admissible Q_3, because (i) imposing $Q_3(\alpha';\underline{\boldsymbol{\alpha}}) = Q^*$ does not reduce the gap value (Lemma IA.8), and (ii) conditional on that restriction, $Q^{\text{lull}}(\cdot)$ maximizes the continuation value from τ_3 onward.

Using Proposition 5, we can obtain the following corollary. The corollary proof relies on continuous dependence as in the proof of Proposition 3.

Corollary (Optimal-policy implementation: Lower and steeper early in the boom). *When the $s=3$ boom begins, under optimal policy with the anticipation effect, the expected policy rate (over horizon h) under any belief $i \in \{o,p,pl\}$ is*

$$\lim_{\hat{\alpha}\to 0} \bar{r}^i_{h,\tau_3,3} = \bar{r}^{\text{lull},i}_{h,\tau_3,3} - \frac{1}{h}\ln\left(\frac{Q^*}{Q^{\text{lull}}(\alpha')}\right).$$

If $Q^{\text{lull}}(\alpha') < Q^*$, then, in the limit as $\hat{\alpha}\to 0$,

$$\bar{r}^i_{h,\tau_3,3} \;<\; \bar{r}^{\text{lull},i}_{h,\tau_3,3} \quad \textit{and} \quad \frac{d}{dh}\bar{r}^i_{h,\tau_3,3} \;>\; \frac{d}{dh}\bar{r}^{\text{lull},i}_{h,\tau_3,3}.$$

That is, the expected policy rate curve is strictly lower and strictly steeper than under the single-cycle optimal rule that ignores the anticipation channel.

Quantitative analysis: Anticipation effect—Simple (non-optimal) leaning rule. Figure IA.8 illustrates the anticipation effect for a simple (non-optimal) leaning rule using our baseline calibration.[21] We consider Q-targeting rules of the form Eq. (28), with $Q_3(\alpha;\underline{\boldsymbol{\alpha}}) = \psi Q_4(\alpha) + (1-\psi)Q^*$, and we set $\psi = 0.5$. The solid blue lines correspond to leaning, under the assumption that the bust arrival rate is exogenous, as in the benchmark model. The dashed blue lines correspond to the same leaning Q-targeting rule, if we assume that this simple leaning rule reduces the bust arrival rate by 10% under all

[21] We set the transition arrival rates and risk premium parameters to be the same across the two booms ($g_1 = g_3$ and $\lambda^i_{1\to2} = \lambda^i_{3\to4}$) and across the two busts ($g_2 = g_4$ and $\lambda^i_{2\to3} = \lambda^i_{4\to5}$).

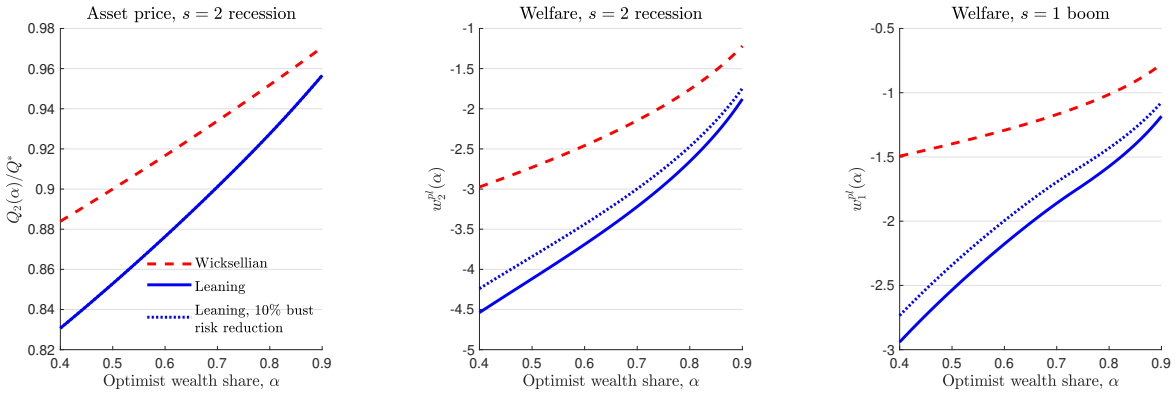


Figure IA.8: **Systematic leaning and the anticipation effect**. This figure illustrates the anticipation effect from a *simple (non-optimal)* leaning rule. The asset price in $s = 2$ (left panel) is identical with an exogenous bust probability (solid blue line) and with a 10% reduction in the bust arrival rate under all beliefs $i \in \{o, p, pl\}$ (dashed blue line), as described further in footnote 22.

belief types $i \in \{o, p, pl\}$. (See Section 5.1 regarding an endogenous bust probability.) The anticipation effect implies a lower asset price during the $s = 2$ recession, under both assumptions about the bust arrival rate, as shown in the left panel.[22] The anticipation effect also implies lower welfare during the $s = 2$ recession, under both the exogenous and the endogenous bust arrival rate scenarios, as shown in the middle panel. When the bust arrival rate is reduced by leaning, a longer expected boom raises welfare, but the anticipation effect, in this scenario, still implies that welfare is lower than under Wicksellian policy. In contrast, as shown later, optimal policy leans and raises welfare, with a larger welfare gain when leaning reduces the bust probability. Moreover, if the simple leaning rule were to reduce the bust arrival rate to a significantly greater extent, then even the simple leaning rule might raise welfare despite the anticipation effect.

We also consider the welfare implications of simple rules from the perspective of the $s = 1$ boom, with the central bank following the same Q-targeting rule in both booms $s \in \{1, 3\}$. Leaning in $s = 3$ reduces welfare through the anticipation effect; this welfare cost will be taken into account—with appropriate discounting—in the $s = 1$ gap value. But leaning also has a welfare benefit: it attenuates the fall in optimists' wealth when the $s = 2$ recession arrives. Taking into account the anticipation effect as well as the other welfare benefits and costs, simple leaning reduces the $s = 1$ gap value, as shown in the

[22] Assumptions about the bust arrival rate do not affect the $s = 2$ asset price, as the boom arrival rate and the Q-targeting rule in $s = 3$ are identical across the two bust-arrival-rate scenarios.

right panel. One reason that simple systematic leaning policies (of the form Eq. (28)) reduce the $s = 1$ gap value is that the anticipation effect, by reducing the asset price in $s = 2$, can exacerbate the fall in optimists' wealth share when the $s = 2$ recession arrives. In this analysis, leaning in $s = 1$ (leaning through $Q_1(\alpha)$, holding $Q_3(\alpha; \underline{\boldsymbol{\alpha}})$ constant) does not generate an anticipation effect because there is no recession prior to $s = 1$ in our two-cycle model. Of course, as more cycles are added, the existence of booms with no preceding recession becomes irrelevant.

Quantitative analysis: Anticipation effect—Optimal policy. Figure IA.9 displays optimal leaning policy with the anticipation effect and its implications for welfare and the recessionary asset price. The blue line, top-left panel, shows (an approximation to) the optimal Q-targeting rule in the second boom, $s = 3$, conditional on a pre-boom optimist wealth share $\underline{\boldsymbol{\alpha}} = 0.8$. The optimal rule involves phased-in leaning, as in the case with optimal unexpected leaning with the announcement effect. With a 10% reduction in bust risk, the strength of optimal leaning is slightly moderated; a longer expected boom raises the welfare costs of below-potential boom-time output. Because the optimal Q-targeting rule sets the asset price to Q^* at the start of the $s = 3$ boom, this rule induces an identical asset price in $s = 2$ as would obtain if the central bank were expected to follow Wicksellian policy in the subsequent boom. This result is illustrated in the top-right panel. During $s = 1$, the optimal Q-targeting rule leans, as shown by the blue line in the bottom-left panel. This optimal $s = 1$ rule does not involve phased-in leaning, as $s = 1$ is the initial state of the $S = \{1, 2, 3, 4, 5\}$ economy; phased-in leaning can insulate a preceding bust from the anticipation effect, but there is no bust that precedes $s = 1$. (As noted above, as more business cycles are added, the existence of booms with no preceding recession becomes irrelevant.) The bottom-right panel shows that optimal leaning raises welfare, and more so, if the bust arrival rate declines as a result of leaning.

Recurrent boom-bust cycles and the anticipation effect

Here, we investigate recurrent cycles (an infinite series of boom-bust cycles). We also discuss what happens if pessimists (or optimists) prevail.

Recurrent cycles: Overview. We assume that there are two states $s \in S$, where $S \equiv \{1, 2\}$, with $\lambda^i_{1\to 2} > 0$ and $\lambda^i_{2\to 1} > 0$. As before, state 1 is a boom with g_1 high, and state

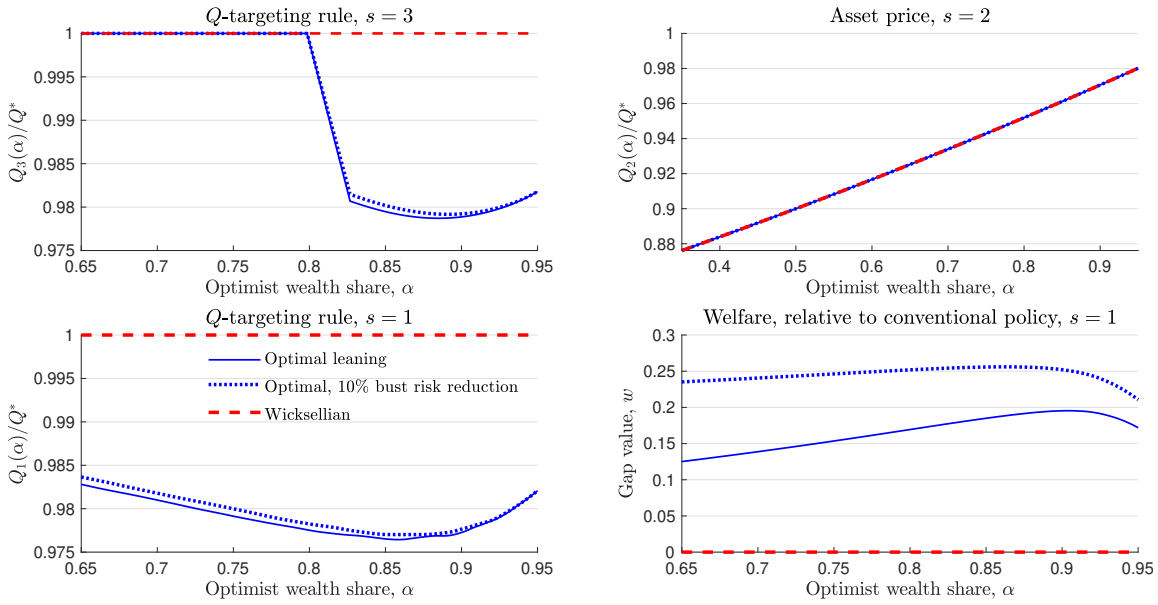


Figure IA.9: **Optimal leaning with the anticipation effect**.

2 is a bust with g_2 low. The bust asset price $Q_2(\alpha)$ satisfies

$$\rho + g_2 - \delta\left(\frac{Q_2(\alpha)}{Q^*}\eta^*\right) - \frac{Q_2'(\alpha)}{Q_2(\alpha)}\alpha(1-\alpha)(\lambda^o_{2\to1} - \lambda^p_{2\to1}) + \bar{\lambda}_{2\to1}(\alpha)\left(1 - \frac{Q_2(\alpha)}{Q_1(\alpha_{2\to1}(\alpha);\alpha)}\right) = 0, \tag{IA.92}$$

where $Q_1(\alpha, \underline{\boldsymbol{\alpha}})$ is the boom Q-targeting rule conditional on optimist wealth share α during the boom and optimist wealth share $\underline{\boldsymbol{\alpha}}$ at the end of the preceding bust. From Eq. (IA.92), with recurrent cycles, the core mechanism of the anticipation effect remains intact: More aggressive leaning during the boom (lower values for $Q_1(\cdot)$, in the sense described in Lemmas IA.6 and IA.7) implies a lower asset price during the bust. This mechanism suggests a policy design analogous to the approach with two cycles: set the asset price equal to Q^* as each boom begins and subsequently phase in leaning.

Recurrent cycles: Equilibrium. *Bust asset price.* Appendix A.4, which nests both the finite-cycle and recurrent-cycle versions of the model, shows the existence and uniqueness of the solution $Q_2(\alpha)$ for Eq. (IA.92) and demonstrates properties of $Q_2(\alpha)$ (Lemma IA.1).

Gap value. With recurrent cycles, the gap value is a pair of functions, $\{w_1^{pl}(\alpha; \underline{\boldsymbol{\alpha}}), w_2^{pl}(\alpha)\}$, where $\underline{\boldsymbol{\alpha}}$ is the optimists' wealth share at the end of the preceding bust. The $s = 1$ gap value depends on $\underline{\boldsymbol{\alpha}}$ because the Q-targeting rule $Q_1(\alpha; \underline{\boldsymbol{\alpha}})$ may depend on $\underline{\boldsymbol{\alpha}}$. The $s = 2$

gap value satisfies

$$\rho w_2^{pl}(\alpha) = W(Q_2(\alpha)) + \lambda_{2\to1}^{pl}\left(w_1^{pl}(\alpha_{2\to1}(\alpha);\alpha) - w_2^{pl}(\alpha)\right) + \frac{\partial w_2^{pl}(\alpha)}{\partial\alpha}\dot{\alpha}_2(\alpha). \tag{IA.93}$$

where $\dot{\alpha}_2(\alpha) = -\alpha(\lambda_{2\to1}^o - \bar{\lambda}_{2\to1}(\alpha))$ and $\alpha_{2\to1}(\alpha) = \alpha\lambda_{2\to1}^o/\bar{\lambda}_{2\to1}(\alpha)$, from Eq. (16), and $Q_2(\alpha)$ is the solution to Eq. (IA.92). The boom gap value satisfies:

$$\rho w_1^{pl}(\alpha;\underline{\boldsymbol{\alpha}}) = W(Q_1(\alpha;\underline{\boldsymbol{\alpha}})) + \lambda_{1\to2}^{pl}\left(w_2^{pl}(\alpha_{1\to2}(\alpha;\underline{\boldsymbol{\alpha}})) - w_1^{pl}(\alpha;\underline{\boldsymbol{\alpha}})\right) + \frac{\partial w_1^{pl}(\alpha;\underline{\boldsymbol{\alpha}})}{\partial\alpha}\dot{\alpha}_1(\alpha;\underline{\boldsymbol{\alpha}}), \tag{IA.94}$$

where

$$\frac{\alpha_{1\to2}(\alpha;\underline{\boldsymbol{\alpha}})}{\alpha} - 1 = (\omega_1^o(\alpha;\underline{\boldsymbol{\alpha}}) - 1)\left(1 - \frac{Q_1(\alpha;\underline{\boldsymbol{\alpha}})}{Q_2(\alpha_{1\to2}(\alpha;\underline{\boldsymbol{\alpha}}))}\right) \tag{IA.95}$$

and

$$\frac{\dot{\alpha}_1(\alpha;\underline{\boldsymbol{\alpha}})}{\alpha} = \lambda_{1\to2}^p\frac{1-\alpha}{1-\alpha_{1\to2}(\alpha;\underline{\boldsymbol{\alpha}})}\left(1 - \frac{\alpha_{1\to2}(\alpha;\underline{\boldsymbol{\alpha}})}{\alpha}\right). \tag{IA.96}$$

Setting $\alpha \in \{0,1\}$ in Eq. (IA.92), in an economy in which all wealth is held by agents with belief $i \in \{o,p\}$, the bust asset price Q_2^i satisfies

$$\rho + g_2 - \delta(\frac{Q_2^i}{Q^*}\eta^*) + \lambda_{2\to1}^i\left(1 - \frac{Q_2^i}{Q_1^i}\right) = 0, \tag{IA.97}$$

where: $Q_1^o = Q_1(1,1)$ and $Q_1^p = Q_1(0,0)$ are the $s=1$ asset price in the all-optimist and all-pessimist economies; and $Q_2^o = Q_2(1)$ and $Q_2^p = Q_2(0)$ are the $s=2$ asset price in these economies. For $\alpha \in \{0,1\}$, the boom gap value is

$$w_1^{pl}(\alpha) = \frac{1}{\rho}\frac{W(Q_1(\alpha,\alpha)) + \frac{\lambda_{1\to2}^{pl}}{\rho+\lambda_{2\to1}^{pl}}W(Q_2(\alpha))}{1 + \frac{\lambda_{1\to2}^{pl}}{\rho+\lambda_{2\to1}^{pl}}} \tag{IA.98}$$

and the bust gap value is

$$w_2^{pl}(\alpha) = \frac{1}{\rho}\frac{W(Q_2(\alpha)) + \frac{\lambda_{2\to1}^{pl}}{\rho+\lambda_{1\to2}^{pl}}W(Q_1(\alpha,\alpha))}{1 + \frac{\lambda_{2\to1}^{pl}}{\rho+\lambda_{1\to2}^{pl}}}. \tag{IA.99}$$

The gap value $(w_1^{pl}(\alpha,\underline{\boldsymbol{\alpha}}), w_2^{pl}(\alpha))$ is the solution to Eqs. (IA.93)–(IA.94), subject to the boundary condition Eqs. (IA.98)–(IA.99).

Denote the economy's initial condition, the optimists' wealth share at $t=0$, by $\alpha_{t=0}$.

Suppose that the economy begins in a boom state ($s = 1$).[23] The planner's problem is:[24]

$$\sup_{\{Q_1(\alpha;\underline{\boldsymbol{\alpha}})\}_{(\alpha,\underline{\boldsymbol{\alpha}})\in[0,1]\times[0,1]}\in\mathcal{C}_r} w_1^{pl}(\alpha_{t=0}) \tag{IA.100}$$

subject to the risk-balance condition (Eq. (IA.92)), the law of motion for optimists' wealth share upon transition from $s = 1$ to $s' = 2$ (Eq. (IA.95)), the evolution of optimists' wealth share absent transition in $s = 1$ (Eq. (IA.96)), and, for $s = 2$, the change in wealth share upon transition and optimists' wealth share growth (Eqs. (13) and (15), respectively).

We refer to Eq. (IA.92) as the sowing-the-wind anticipation channel. The recurrent-cycle equations nevertheless motivate a direct analogue of the two-cycle phased-in policy. Let $\hat{Q}_1(\alpha)$ denote the rule obtained when the anticipation channel for Eq. (IA.100) is replaced by the single-cycle risk-balance condition, Eq. (19). For $\alpha' = \alpha_{2\to1}(\underline{\boldsymbol{\alpha}})$, the corresponding recurrent phase-in rule sets

$$Q_1(\alpha';\underline{\boldsymbol{\alpha}}) = Q^*$$

and subsequently transitions to $\hat{Q}_1(\alpha)$. If $\hat{Q}_1(\alpha') = Q^*$, no transition is required; if $\hat{Q}_1(\alpha') < Q^*$, the transition can be implemented by a linear ramp analogous to Eqs. (42)–(43). Setting the asset price equal to Q^* at boom arrival removes the direct anticipation channel in Eq. (IA.92). The recurrent economy therefore features the same economic force favoring phased-in leaning as the two-cycle economy.

Remark—When pessimists (or optimists) prevail. Belief differences among investors—and the resulting speculation in financial markets—are necessary for there to be any potential welfare benefit from leaning policies. Specifically, in an economy in which pessimists have prevailed ($\alpha = 0$) or optimists have done so ($\alpha = 1$), a lower asset price during the boom reduces utilization during the boom (from Eq. (9), which does not depend on belief differences) without increasing utilization during subsequent recessions.[25]

[23] This assumption is without loss of generality. The optimal policies do not depend on the initial optimist wealth share or initial state $s \in \{1, 2\}$.

[24] We write $Q_1(\alpha;\underline{\boldsymbol{\alpha}}) \in \mathcal{C}_r$ to mean Q_1 is jointly continuous on $[0,1]^2$ and right-differentiable in each argument.

[25] When the economy's initial condition features a non-degenerate mix of optimists and pessimists, with $\alpha_{t=0} \in (0,1)$, pessimists prevailing (with $\alpha_{t,2} = 0$) is an infrequent occurrence, for the following reasons. During busts, optimists' wealth share declines over time: $\dot{\alpha}_2(\alpha) < 0$. However, $\dot{\alpha}_2(\alpha)$ converges to 0 as α approaches 0, from Eq. (16). Moreover, the arrival of the subsequent boom increases optimists' wealth share, which continues to grow as long as the boom continues. (A similar argument applies to optimists prevailing.)

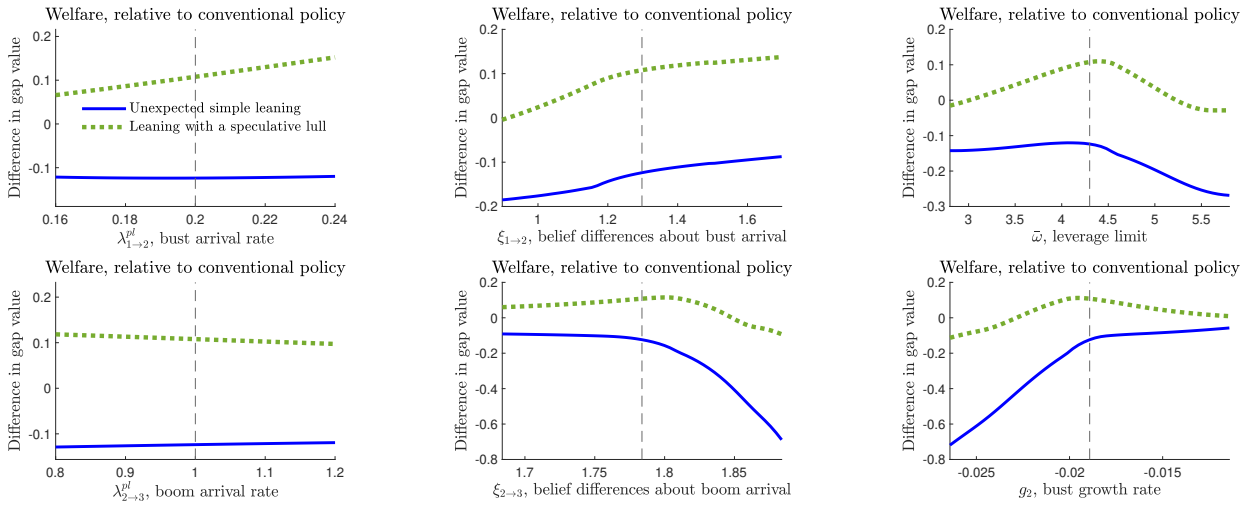


Figure IA.10: **Sensitivity analyses: Welfare implications of a simple, non-optimal leaning policy**. Each panel shows the change in gap value when a simple (non-optimal) leaning rule is announced during a lull (dotted green line) and unexpectedly (solid blue line), varying the value of the indicated parameter. The leaning policy is of the form Eq. (28) with $\psi = 0.5$. The calibrated parameter value is shown by a dashed gray vertical line.

Therefore, the optimal boom-state Q-targeting rule features $Q_1(\alpha) = Q^*$ for $\alpha \in \{0, 1\}$. For intermediate values of the optimist wealth share ($\alpha \in (0, 1)$), the implications of alternative monetary policies depend on how these policies affect the stochastic evolution over time of optimists' wealth share. For example, as discussed in Section 2, leaning reduces optimists' wealth share gains during a boom but cushions the fall in optimists' wealth share upon recession. Optimal Q-targeting rules fully take these considerations into account, as shown by the constraint Eq. (34), for example.

E.4 Sensitivity analysis

We now present analyses of how parameter values affect the welfare implications of different leaning policies. Main-text Figure 4, left panel, shows that a simple, non-optimal leaning rule can reduce welfare if announced unexpectedly. The rule is simple in that it immediately reduces the asset price upon announcement. Each panel in Figure IA.10 repeats the welfare analysis in main-text Figure 4, left panel, for $\psi = 0.5$ and varying the indicated parameter value. The solid blue line shows the change in welfare from unexpected simple leaning. The parameter value in the benchmark parametrization is marked by a dashed vertical line. We vary the bust arrival rate $\lambda^{pl}_{1\to2}$ (top-left panel), the extent of belief differences regarding bust arrival $\xi_{1\to2}$ (top middle, see also Eq. (23)), and

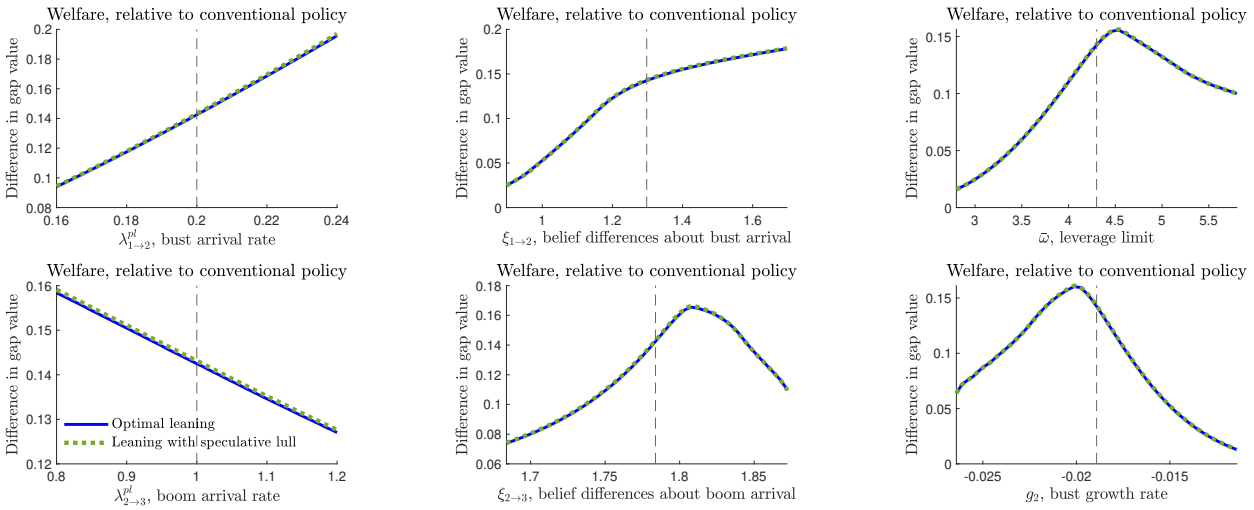


Figure IA.11: **Sensitivity analyses: Welfare implications of optimal leaning**. The solid blue line shows the gap value, relative to Wicksellian policy, under the smoothed approximation (Eq. (35)) to optimal policy. The dotted green line shows the gap value, relative to Wicksellian policy, when the optimal policy with a lull is announced during a lull.

the leverage limit $\bar{\omega}$ (top right).[26] We also vary the recovery arrival rate $\lambda^{pl}_{2\to3}$ (bottom left), the extent of belief differences regarding recovery arrival $\xi_{2\to3}$ (bottom middle), and the bust growth rate g_2 (bottom right). Across a range of parameter values, the unexpected announcement of this simple leaning rule reduces welfare, due to the announcement effect.

By contrast, main-text Figure 5, top-right panel, shows that optimal leaning increases welfare relative to Wicksellian policy. Figure IA.11 repeats this analysis, for $\bar{\alpha} = 0.8$. Across a range of parameter values, optimal leaning increases welfare and achieves nearly the same welfare benefit as if the central bank benefited from announcing during a speculative lull. The welfare gains from optimal policy have an inverted-U relation with the leverage limit—consistent with the intuition in Caballero and Simsek (2020a) that using monetary policy to lean is unnecessary when regulation has already dampened speculation (low leverage limit) and is ineffective when the leverage limit is very high.

[26] We increase $\xi_{1\to2}$ by lowering $\lambda^{o}_{1\to2}$ and raising $\lambda^{p}_{1\to2}$ by the same amount. We set $\bar{\alpha} = 0.8$.

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